2.6 Limits at Infinity; Horizontal Asymptotes

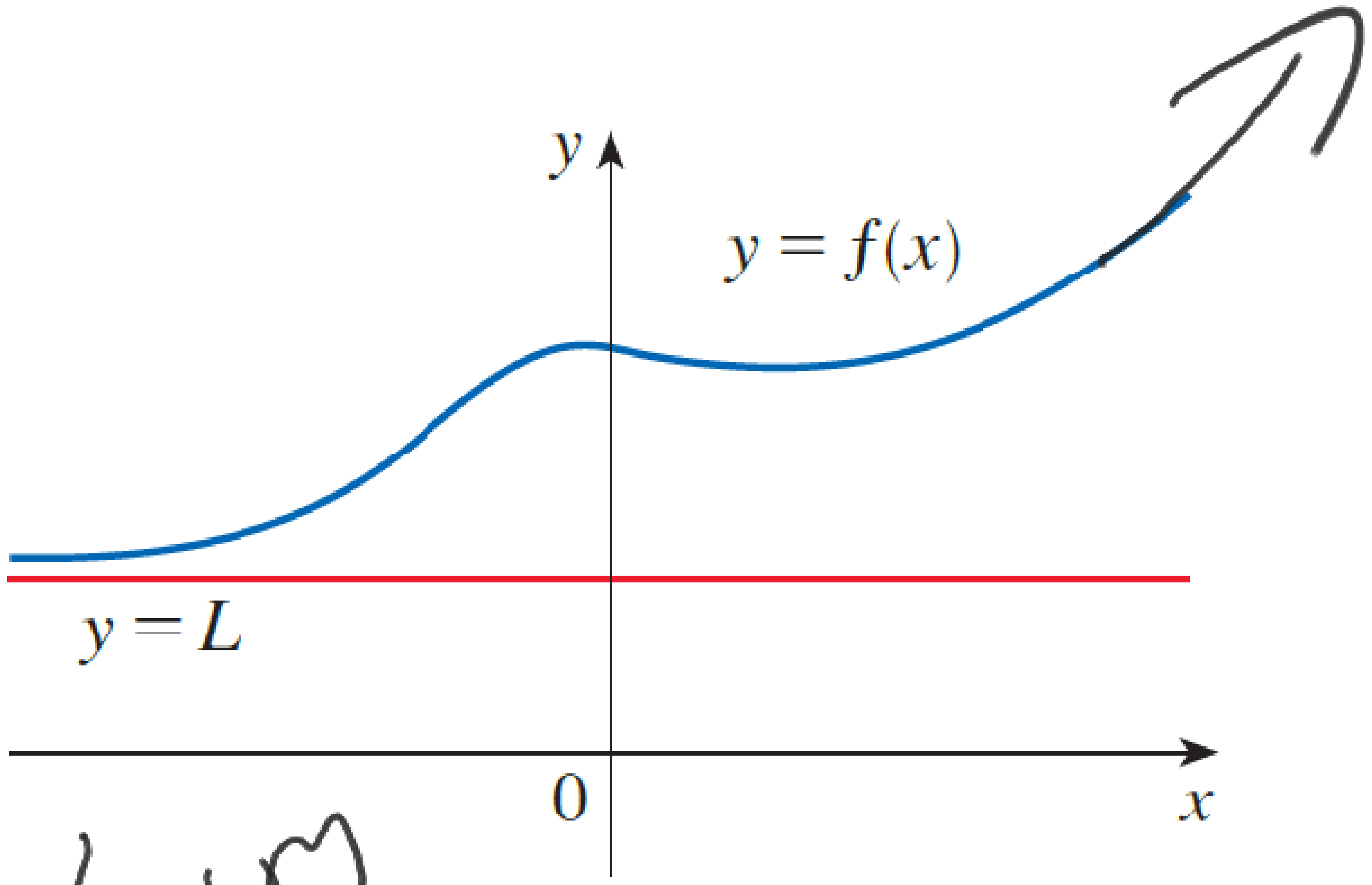
$$f(x) = \frac{3x}{x^2+5} \qquad HA \quad y = 0$$

$$g(x) = \frac{7x^2}{x-2} \qquad HA \quad \text{None}$$

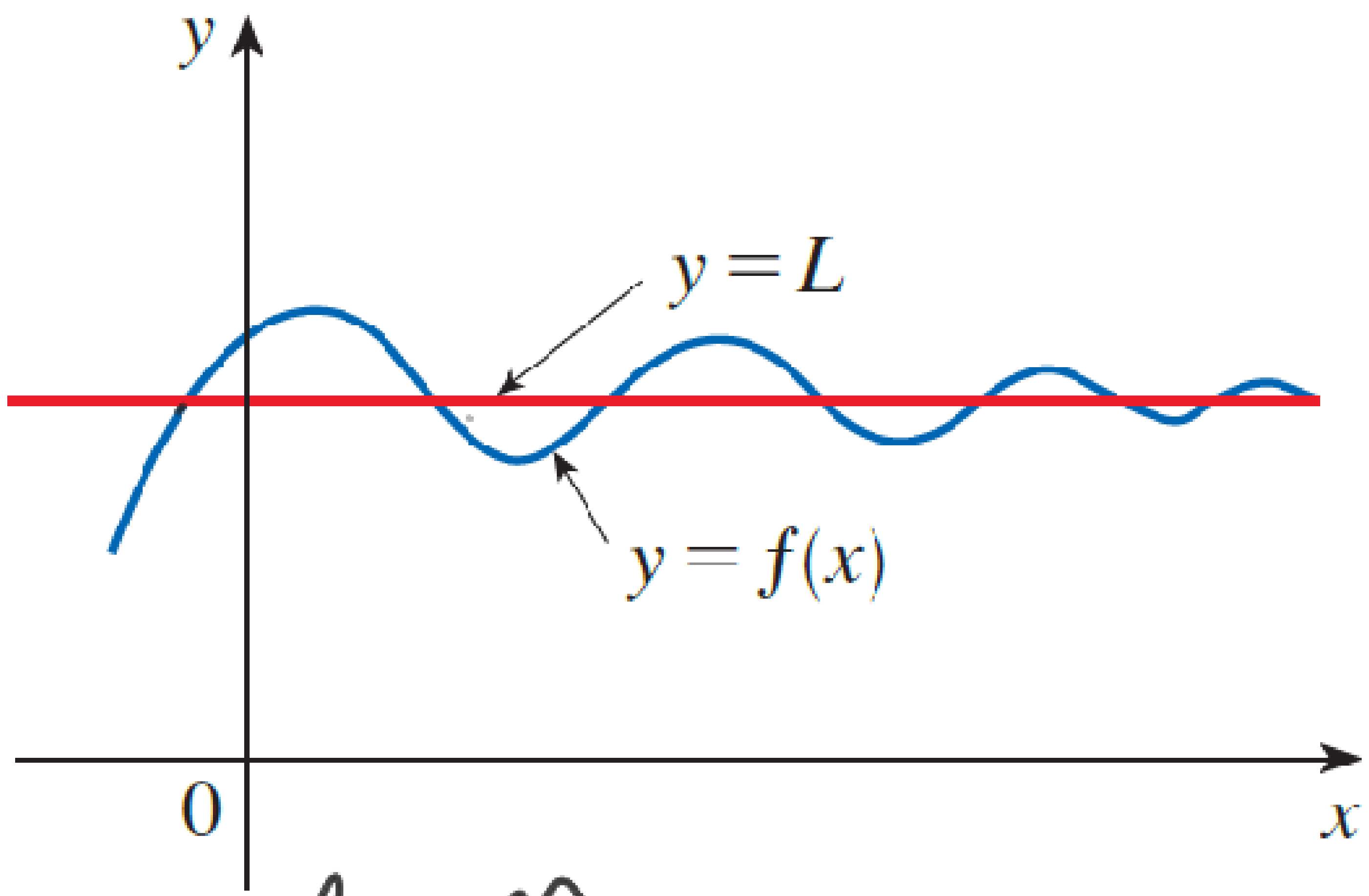
$$h(x) = \frac{5x^3-2x+7}{6x^3+5x^2-2} \qquad HA = \frac{5}{6}$$

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$



$\lim_{x \to -\infty}$



$\lim_{x \to \infty}$

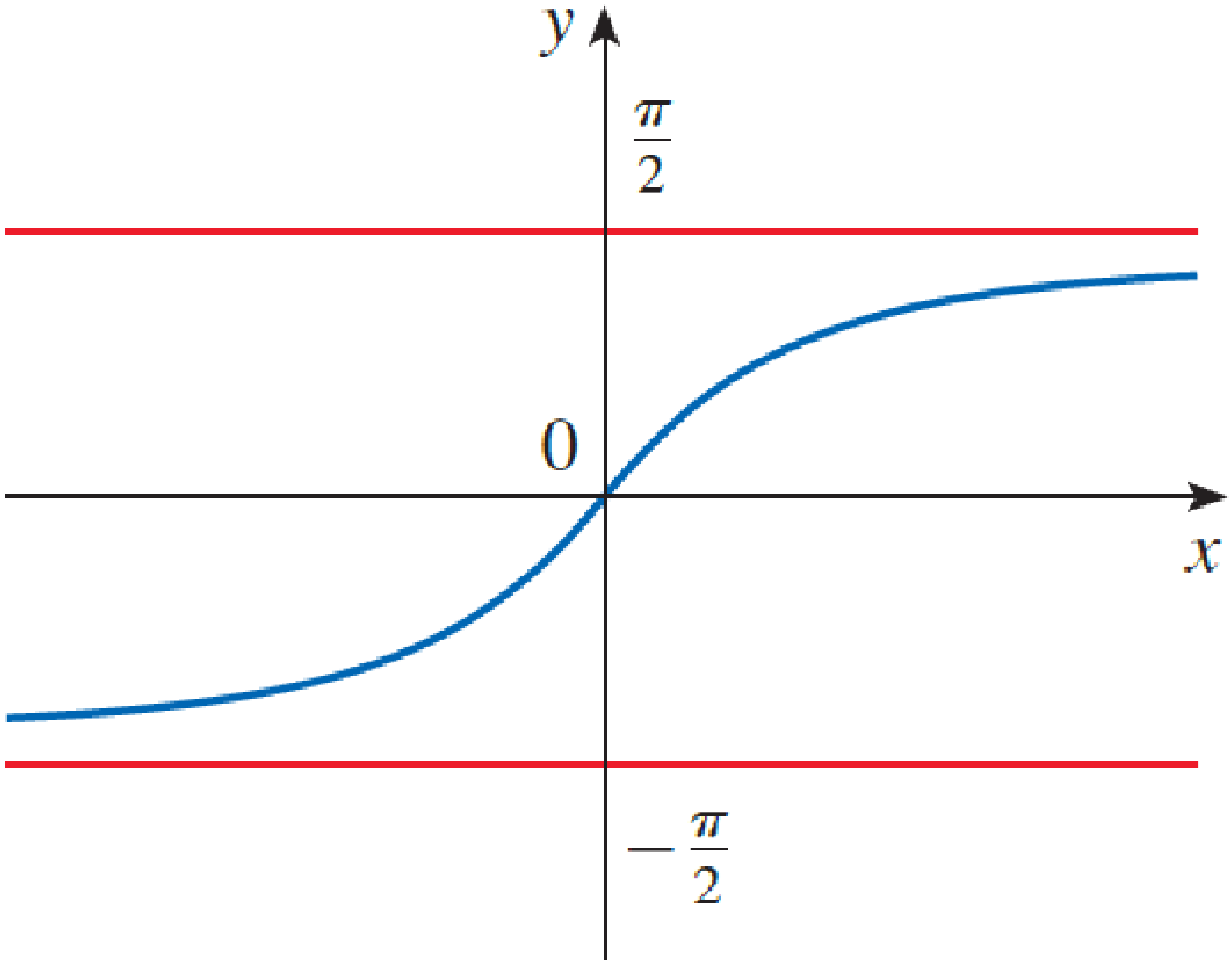
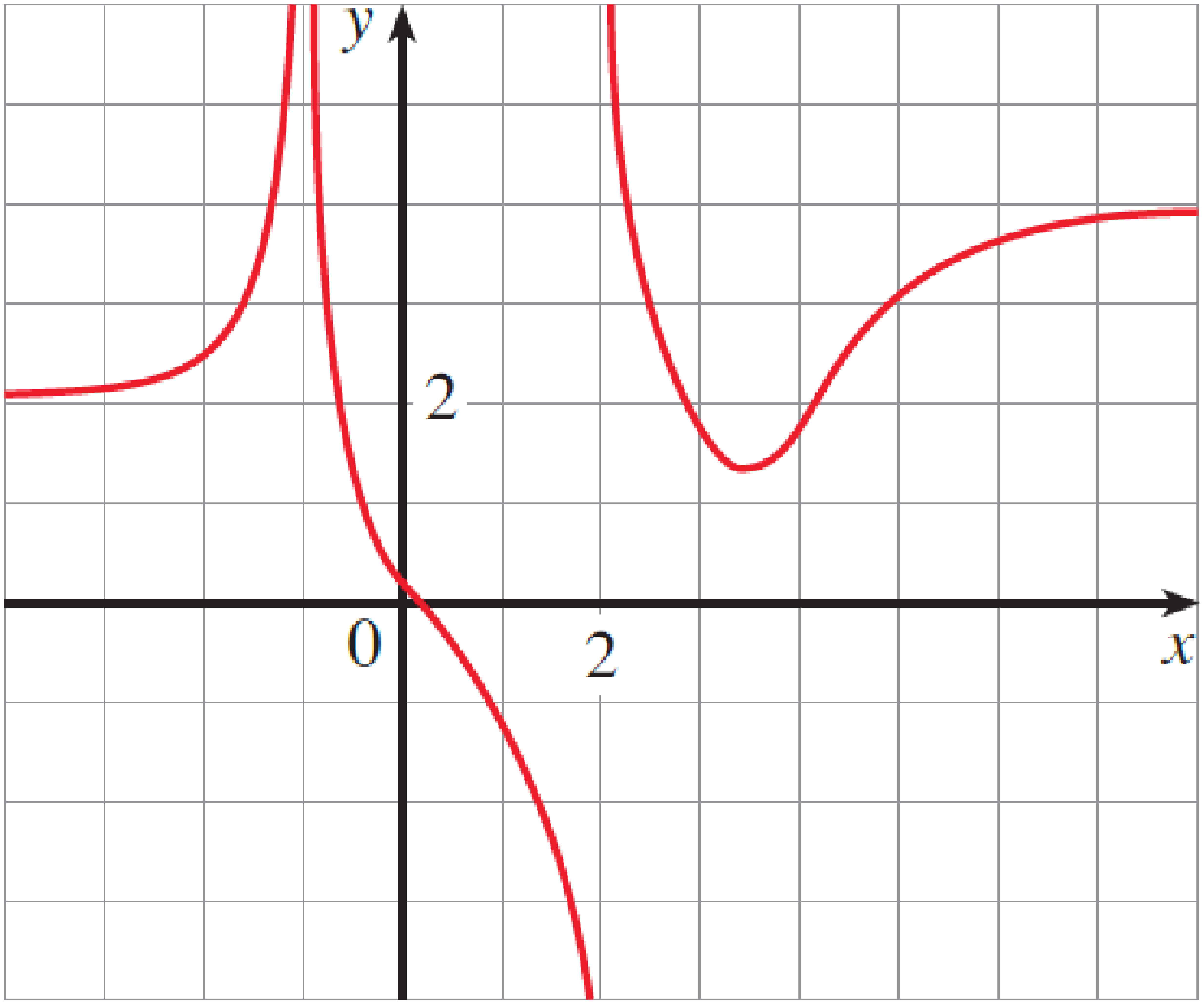


FIGURE 4
$y = \tan^{-1}x$

$$\lim_{x \to -\infty} f(x) = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} f(x) = \frac{\pi}{2}$$



V.A.

$x = -1$

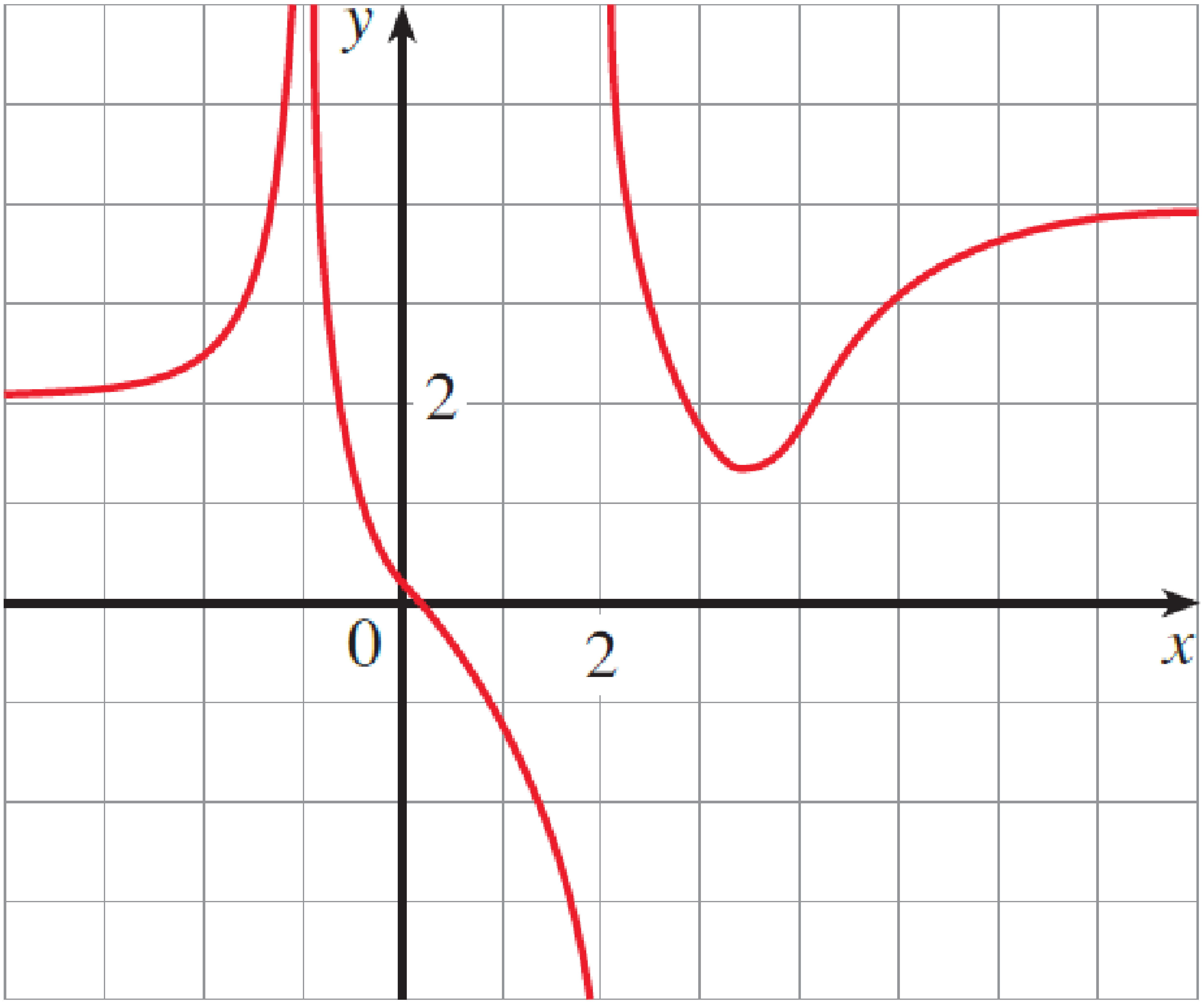
$\lim_{x \to -1} f(x) = \infty$

$x = 2$

$\lim_{x \to 2} f(x) = \text{DNE}$

$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2^+} f(x) = \infty$



HA

$\lim_{x \to -\infty} f(x) = 2$

$\lim_{x \to \infty} f(x) = 4$

$y = 2$
$y = 4$

Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$\boxed{5}$ **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$\frac{\frac{3x^2 - x - 2}{x^2}}{\frac{5x^2 + 4x + 1}{x^2}}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} = \boxed{\frac{3}{5}}$$

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}\,/x}{3x-5\,/x} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{2x^2+1}}{x}}{\dfrac{3x-5}{x}} \rightarrow 3-\frac{5}{x}$$

$$\frac{\sqrt{2x^2+1}}{\sqrt{x^2}} = \sqrt{\frac{2x^2+1}{x^2}} = \sqrt{2+\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{3-5/x} = \boxed{\frac{\sqrt{2}}{3}}$$

Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$. $\boxed{\left(\sqrt{x^2+1}-x\right)\cdot\left(\sqrt{x^2+1}+x\right)}$

$$\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)}$$

$$\left(\sqrt{x^2+1}\right)^2-x^2$$

$$x^2+1-x^2$$

$$1$$

$$\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$$

Evaluate $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$

$$\arctan\left(\boxed{\lim_{x\to 2^+}\left(\frac{1}{x-2}\right)}\right) \neq \arctan(\infty)$$

$$\to \infty$$

$$\boxed{\lim_{m\to\infty} \arctan(m)} = \frac{\pi}{2}$$

$$\lim_{x \to \infty} \arctan\left(\frac{1}{x-2}\right)$$

$$\arctan\left(\lim_{x \to \infty} \frac{1}{x-2}\right)$$

$$\arctan(0) = 0$$

Evaluate $\lim_{x \to \infty} \sin x$. = DNE

Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

$$\lim_{x \to \infty} x^3 = \infty$$

$$\lim_{x \to -\infty} x^3 = -\infty$$

Find $\lim_{x \to \infty} (x^2 - x)$. $= \infty$

$$\lim_{x \to \infty} x(x - 1) = \infty$$

Find $\lim\limits_{x\to\infty} \dfrac{x^2+x}{3-x}$. $\dfrac{/x}{/x} = \dfrac{x+1}{3/x - 1} = \dfrac{\infty}{-1} = -\infty$

0

$$\frac{1 + 1/x}{3/x^2 - 1/x} = \frac{1}{0}$$

$$\lim_{\to -\infty} \frac{\left(u^2+1\right)\left(2u^2-1\right)}{\left(u^2+2\right)^2} = \lim_{u\to-\infty} \frac{\left[\left(u^2+1\right)\left(2u^2-1\right)\right]/u^4}{\left(u^2+2\right)^2/u^4} = \lim_{u\to-\infty} \frac{\left[\left(u^2+1\right)/u^2\right]\left[(2u^2-1)/u^2\right]}{\left(u^4+4u^2+4\right)/u^4}$$

$$= \lim_{u\to-\infty} \frac{\left(1+1/u^2\right)\left(2-1/u^2\right)}{\left(1+4/u^2+4/u^4\right)} = \frac{(1+0)(2-0)}{1+0+0} = 2$$

$$\lim_{x\to\infty} \frac{4-\sqrt{x}}{2+\sqrt{x}} = \lim_{x\to\infty} \frac{\left(4-\sqrt{x}\right)/\sqrt{x}}{\left(2+\sqrt{x}\right)/\sqrt{x}} = \lim_{x\to\infty} \frac{4/\sqrt{x}-1}{2/\sqrt{x}+1} = \frac{0-1}{0+1} = -1$$

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$$

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right]=\lim_{x\to-\infty}\frac{(4x^2+3x)-(2x)^2}{\sqrt{4x^2+3x}-2x}$$

$$=\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}\qquad\lim_{x\to-\infty}\frac{3x/x}{\left(\sqrt{4x^2+3x}-2x\right)/x}$$

$$=\lim_{x\to-\infty}\frac{3}{-\sqrt{4+3/x}-2}$$

$$=\frac{3}{-\sqrt{4+0}-2}=-\frac{3}{4}$$

$$x=\sqrt{x^2}$$

$$x=\pm\sqrt{x^2}$$

$$\lim_{x \to \infty} \left(e^{-2x} \cos x\right) \to 0 \qquad e^{-2x} = \frac{1}{e^{2x}} = 0$$

$$-1 \le \cos x \le 1$$

$$-e^{-2x} \le e^{-2x} \cos x \le e^{-2x}$$

$$0 \le \boxed{e^{-2x} \cos x} \le 0$$