Infinite Sequences

An **infinite sequence**, or just a **sequence**, can be thought of as a list of numbers written in a definite order:

$$a_1, \; a_2, \; a_3, \; a_4, \; \ldots, \; a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

$$\left\{\frac{1}{2^n}\right\} \qquad a_n = \frac{1}{2^n} \qquad \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, \frac{1}{2^n}, \ldots\right\}$$

$$\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} \rightarrow \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

$n=0 \quad n=1 \quad n=2 \quad n=3$

$$\left\{(-1)^n \frac{(n+1)}{3^n}\right\}_{n=0}^{\infty} \rightarrow 1, -\frac{2}{3}, \frac{3}{9}, \frac{-4}{27}, \frac{5}{81}, \ldots$$

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

$$a_n = (-1)^{n+1}\left(\frac{n+2}{5^n}\right)$$

2. $\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\} = e$

1 Intuitive Definition of a Limit of a Sequence A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

$\lim_{n \to \infty} a_n = L$ Converges

$\lim_{n \to \infty} a_n = \infty$ Diverges

$a_n = \sin n\pi \quad \{0, 0, 0, 0, 0, 0, 0, \ldots$

$\lim_{n\to\infty} a_n = 0$

$a_n = \cos n\pi \quad \{-1, 1, -1, 1, -1, \ldots\}$

$\lim_{n\to\infty} a_n \to$ DNE

$$\lim_{n \to \infty} \frac{1}{n^r} = 0$$

Limit Laws for Sequences Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

Sum Law

1. $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$

Difference Law

2. $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n$

Constant Multiple Law

3. $\lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n$

Product Law

4. $\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$

Quotient Law

5. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$ if $\lim_{n \to \infty} b_n \neq 0$

Power Law

$$\lim_{n \to \infty} a_n^p = \left[\lim_{n \to \infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

EXAMPLE 4 Find $\lim_{n\to\infty} \dfrac{n}{n+1}$

Converges
$L = 1$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \boxed{\lim_{n\to\infty} \frac{1}{n}}}$$

$$= \frac{1}{1+0} = 1$$

EXAMPLE 5 Is the sequence $a_n = \dfrac{n}{\sqrt{10 + n}}$ convergent or divergent?

$$\lim_{n\to\infty} \frac{n \;/n}{\sqrt{10+n}\;/n} = \frac{n/n}{\sqrt{10/n^2 + n/n^2}}$$

$$\frac{\sqrt{10+n}}{n} = \frac{\sqrt{10+n}}{\sqrt{n^2}} = \sqrt{\frac{10+n}{n^2}}$$

$$\boxed{\frac{1}{\sqrt{0+0}}} = \boxed{\frac{1}{0}} \to \infty \qquad \frac{\lim_{n\to\infty} 1}{\boxed{\sqrt{\lim_{n\to\infty} 10/n^2 + \lim_{n\to\infty} 1/n}}} = \frac{1}{} = \infty$$

EXAMPLE 6 Calculate $\lim_{n \to \infty} \frac{\ln n}{n}$

$\ln n \Rightarrow 1/n$

$n \to 1$

d/dx

L' Hospital

$$\lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \frac{1/n}{1} = \frac{0}{1} = 0$$

Converges

6 Theorem If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

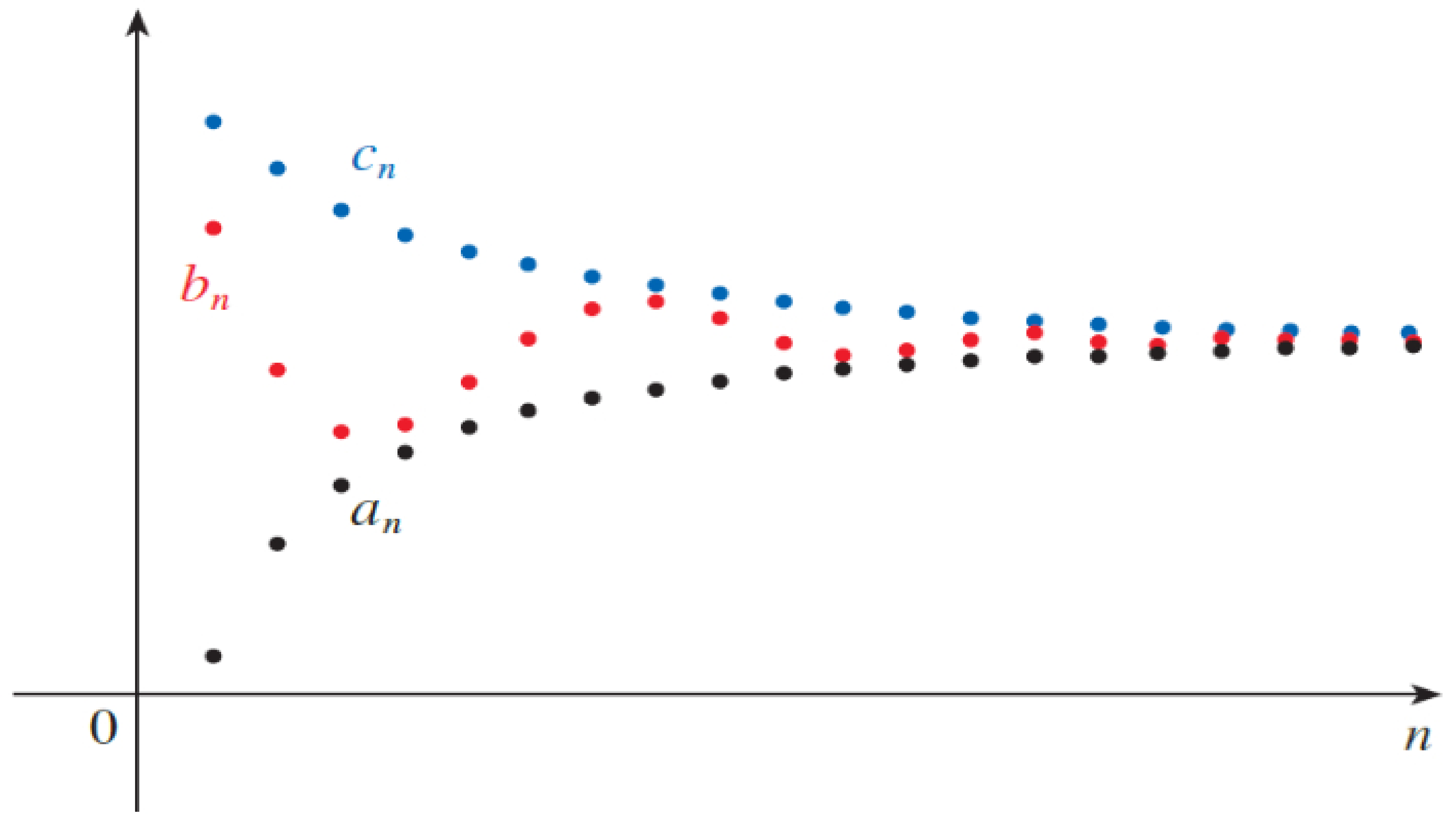
EXAMPLE 8 Evaluate $\lim_{n \to \infty} \frac{(-1)^n}{n}$ if it exists.

$$\lim_{n \to \infty} \left| \frac{(-1)^n}{n} \right| \to \lim_{n \to \infty} \frac{1}{n} \to 0$$

$$\lim_{n \to \infty} \frac{(-1)^n}{n} \to 0$$

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.



EXAMPLE 10 Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

$a_1 = 1 \qquad a_2 = \frac{\cancel{1} \cdot 2}{\cancel{2} \cdot 2} = 1 \qquad a_3 = \frac{\cancel{1} \cdot 2 \cdot 3}{\cancel{3} \cdot 3 \cdot 3} = \frac{6}{9}$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n} \qquad \frac{2 \cdot 3 \cdot 4}{4 \cdot 4 \cdot 4} = \frac{24}{64}$$

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right) \qquad 0 < a_n \leqslant \frac{1}{n}$$

We know that $1/n \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

49. $a_n = \dfrac{\cos^2 n}{2^n}$ $\quad \lim\limits_{n\to\infty} a_n \to 0$

$$\frac{0}{2^n} \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$$

$$0 \le \frac{\cos^2 n}{2^n} \le 0 \qquad \lim_{n\to\infty} \frac{1}{2^n} \to 0$$

$$a_n = \frac{4^n / 9^n}{1 + 9^n / 9^n} \rightarrow \frac{4^n/9^n}{1/9^n + 9^n/9^n}$$

$$\lim_{n\to\infty} \frac{(4/9)^n}{(1/9^n) + 1} \rightarrow \frac{0}{0+1} = 0$$

Converges

$$a_n = 3^n 7^{-n} = \frac{3^n}{7^n}$$

$$\lim_{n\to\infty} \left(\frac{3}{7}\right)^n \to 0$$

$$a_n = \sqrt{\frac{1+4n^2}{1+n^2}} \Big/ n^2 \qquad \lim_{n\to\infty} a_n \quad \text{Converges}$$

$$\sqrt{\lim_{n\to\infty}\left(\frac{1+4n^2}{1+n^2}\right)} \to \sqrt{\lim_{n\to\infty} \frac{1/n^2+4}{1/n^2+1}} \to$$

$$\sqrt{\frac{4}{1}} = \boxed{2}$$

$$a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{\cancel{(2n-1)!}}{(2n+1)(2n)\cancel{(2n-1)!}}$$

$$(2n+1)! = (2n+1)(2n)!$$
$$= (2n+1)(2n)(2n-1)!$$

$$a_n = \frac{1}{(2n+1)(2n)} \to 0$$

$$a_n = \ln(n+1) - \ln n$$

$$= \ln\left(\frac{n+1}{n}\right)$$

$$= \ln\left(1+\frac{1}{n}\right)$$

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)$$

$$\to \ln(1) = 0$$

7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$