$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

$a = 3$

$f(a) = 1$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \to 0} \frac{f(3+h) - (1)}{h}$$

$$\lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h}$$

$$\lim_{h\to 0} \frac{\boxed{\dfrac{3}{3+h} - 1}}{h}$$

$$\frac{3 - 1(3+h)}{3+h}$$

$$\frac{3-3-h}{3+h} = \frac{-h}{3+h}$$

$$\lim_{h\to 0} \frac{\dfrac{-h}{3+h}}{\cancel{h}_{1}}$$

$$\frac{-\cancel{h}}{3+h} \times \frac{1}{\cancel{h}} = \frac{-1}{3+h}$$

$$\lim_{h\to 0} \frac{-1}{3+h} = \boxed{\frac{-1}{3}}$$

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

$f(x) = 4.9t^2$

$a = 5$

$f(5) = 122.5$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \to 0} \frac{4.9(5+h)^2 - 122.5}{h}$$

$4.9(25 + 10h + h^2)$

$122.5 + 49h + 4.9h^2 - 122.5$

$49h + 4.9h^2$

$h(49 + 4.9h)$

$$\lim_{h \to 0} \frac{\cancel{h}(49 + 4.9h)}{\cancel{h}} = \lim_{h \to 0} 49 + 4.9h$$

$$= \boxed{49}$$

$f(x) = 4.9x^2$

$$\lim_{h \to 0} \frac{4.9(a+h)^2 + (4.9a^2)}{h}$$

$$\lim_{h \to 0} 9.8a + 9.8h = \boxed{9.8a}$$

position $4.9t^2$ velocity $9.8t$

acceleration 9.8

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

first derivative

$$f'(a)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\boxed{f'(x) = y'} = \boxed{\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)} = Df(x) = D_x f(x)$$

Newton

Leibniz

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^3 - (x+h)$$

$$(x+h)(x+h)(x+h)$$

$$(x^3 + 3x^2h + 3xh^2 + h^3) - (x+h)$$

$$f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 - x - h$$

$(x+h)(x+h)$

$x^2+hx+hx+h^2$

$(x^2+2hx+h^2)(x+h)$

$x^3+2hx^2+h^2x+x^2h+2h^2x+h^3$

$x^3+3hx^2+3h^2x+h^3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{(\cancel{x^3} + 3x^2 h + 3xh^2 + h^3 \cancel{- x} - h) - (\cancel{x^3} \cancel{- x})}{h}$$

$$\lim_{h \to 0} 3x^2 + 3xh + h^2 - 1 = \boxed{3x^2 - 1}$$

If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$\lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

$$f(x+h) = \frac{1-(x+h)}{2+(x+h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\frac{1-x-h}{2+x+h} - \frac{1-x}{2+x}}{h} = \frac{\frac{-3h}{(2+x+h)(2+x)}}{h}$$

$$\lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)} = \boxed{\frac{-3}{(2+x)^2}}$$

$$\frac{1-x-h}{2+x+h} - \frac{1-x}{2+x}$$

$$\frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)}$$

$$\frac{(2+x-2x-x^2-2h-hx) - (2+x+h-2x-x^2-xh)}{}$$

$$= \frac{-3h}{(2+x+h)(2+x)}$$

Where is the function $f(x) = |x|$ differentiable?

$$\lim_{h \to 0^+} \frac{|x+h| - |x|}{h}$$

$x > 0$

$|x| = x$

$$\lim_{h \to 0^+} \frac{(x+h) - (x)}{h}$$

$$\lim_{h \to 0^+} \frac{h}{h} = 1$$

Where is the function $f(x) = |x|$ differentiable?

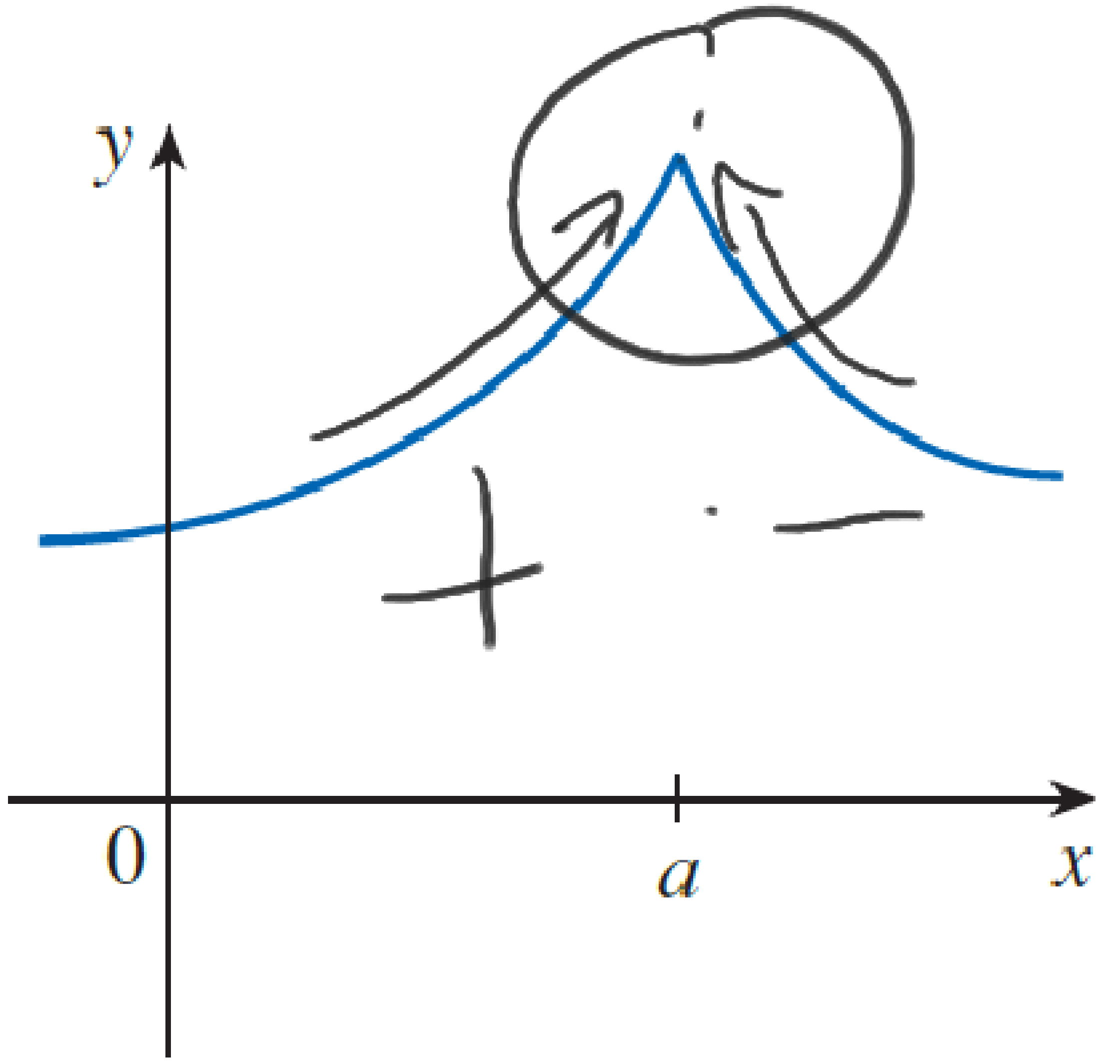
$$\lim_{h \to 0^-} \frac{|x+h| - |x|}{h}$$

$x < 0$

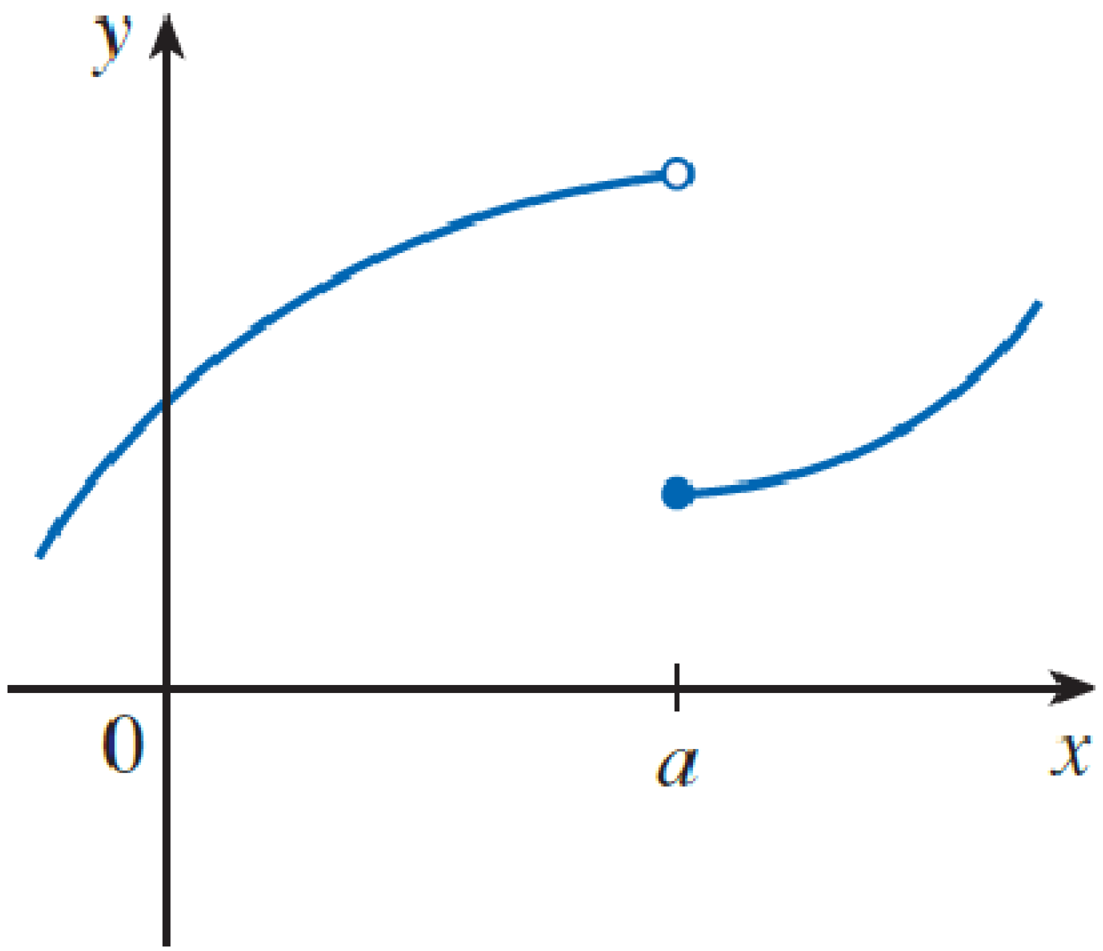
$|x| = -x$

$$\lim_{h \to 0^-} \frac{-(x+h) - (-x)}{h}$$

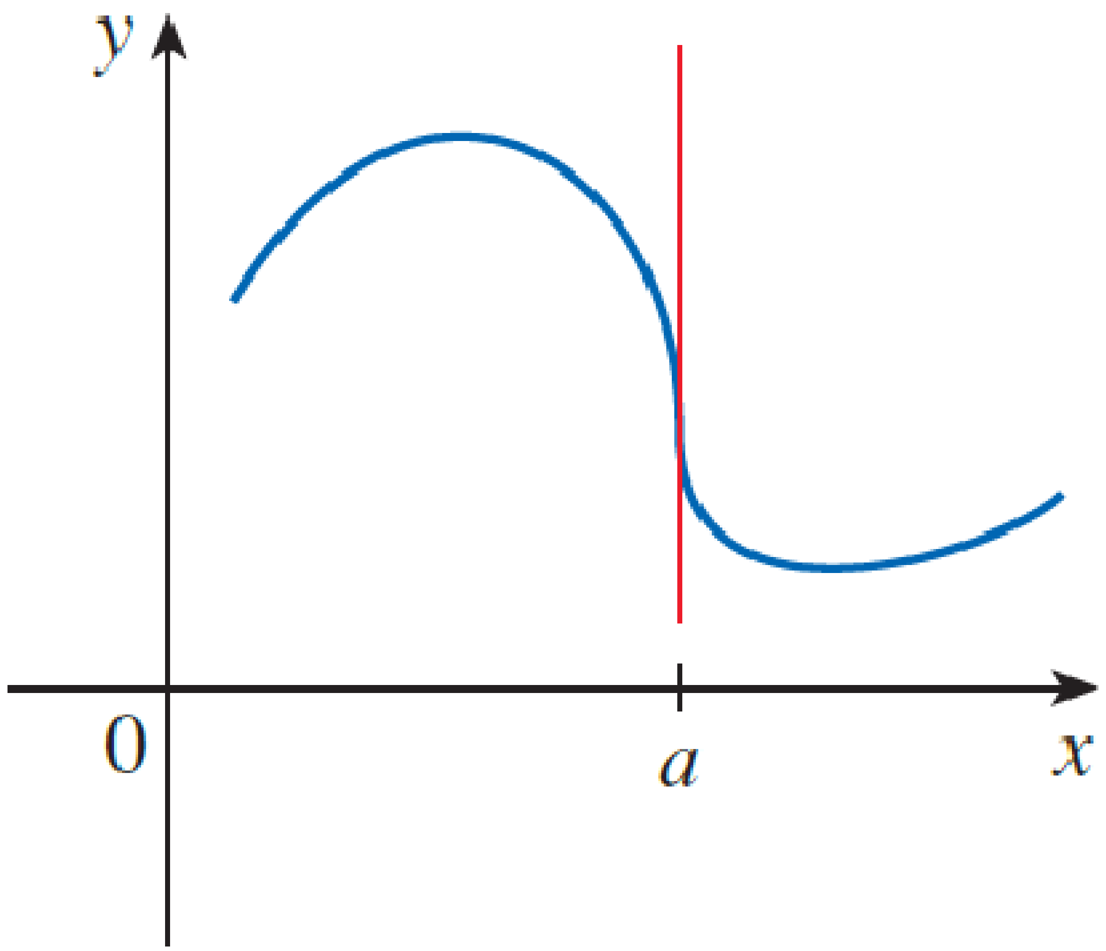
$$\lim_{h \to 0^-} \frac{-h}{h} = -1$$



(a) A corner



(b) A discontinuity



(c) A vertical tangent

$f(x)$	$\rightarrow$ position	function
$f'(x)$	$\rightarrow$ velocity	first derivative
$f''(x)$	$\rightarrow$ acceleration	second derivative
$f'''(x)$	$\rightarrow$ jerk	third derivative

The height (in meters) of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h = 2 + 24.5t - 4.9t^2$ after t seconds.

(a) Find the velocity after 2 s and after 4 s.

(b) When does the projectile reach its maximum height?

(c) What is the maximum height?

(d) When does it hit the ground?

(e) With what velocity does it hit the ground?

$f'(t) = 24.5 - 9.8t = v(t)$

$f(t) = 2 + 24.5t - 4.9t^2$

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$f(t+h) = 2 + 24.5(t+h) - 4.9(t+h)^2$