What Does f' Say about f?

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$$f'(x) = 12x^3 - 12x^2 - 24x \qquad x = 0$$

$$0 = 12x(x^2 - x - 2) \qquad x = 2$$

$$0 = 12x(x-2)(x+1) \qquad x = -1$$

$(-\infty, -1)$	-2	$(-)(-)(-) = -$	Decressing
$(-1, 0)$	$-1/2$	$(-)(-)(+) = +$	Incressing
$(0, 2)$	1	$(+)(-)(+) = -$	Decressing
$(2, \infty)$	5	$(+)(+)(+) = +$	Incressing

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right ot c, or negative to the left and right of c, then f has no local maximum or minimum at c.

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$0 = 12x(x^2 - x - 2)$$

$$0 = 12x(x-2)(x+1)$$

$x = 0$ max

$x = 2$ min

$x = -1$ min

$(-\infty, -1)$	-2	$(-)(-)(-) = -$	Decressing
$(-1, 0)$	$-1/2$	$(-)(-)(+) = +$	Incressing
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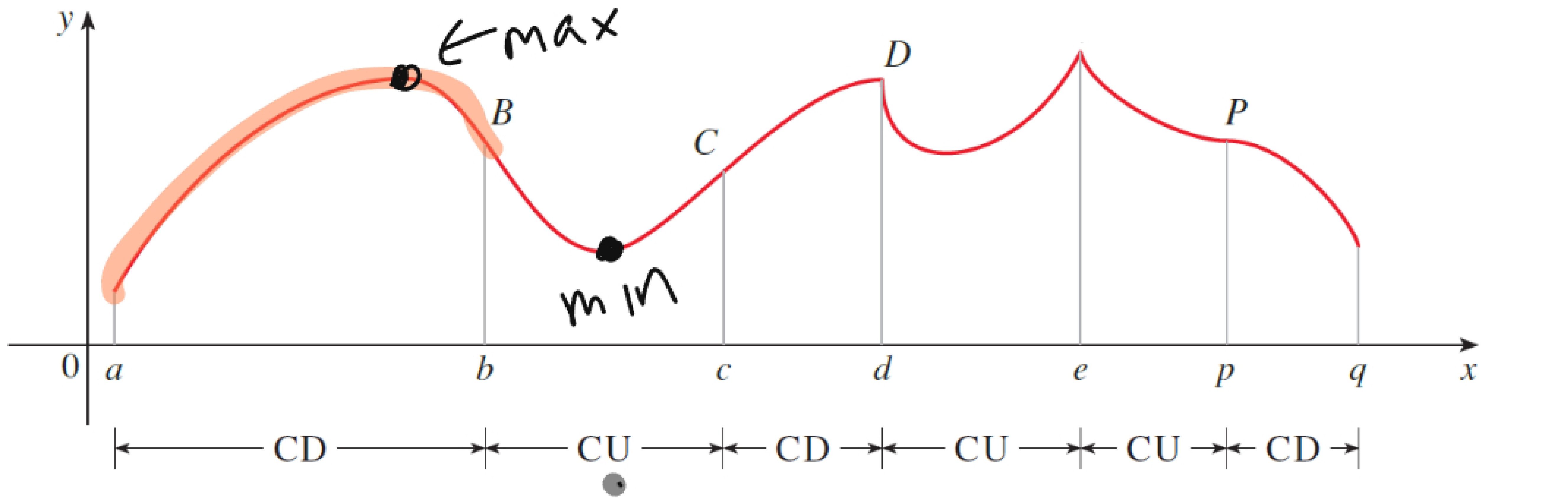
What Does f'' Say about f?

Concavity Test

(a) If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.



The Second Derivative Test

Another application of the second derivative is the following test for identifying local maximum and minimum values. It is a consequence of the Concavity Test, and it serves as an alternative to the First Derivative Test.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

$g'(x) = 1 + 2\cos x$

$0 = 1 + 2\cos x$

$-\frac{1}{2} = \cos x$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$g''(x) = -2\sin x$

$x = \frac{2\pi}{3} \quad g''\left(\frac{2\pi}{3}\right) = -$ max

$x = \frac{4\pi}{3} \quad g''\left(\frac{4\pi}{3}\right) = +$ min

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

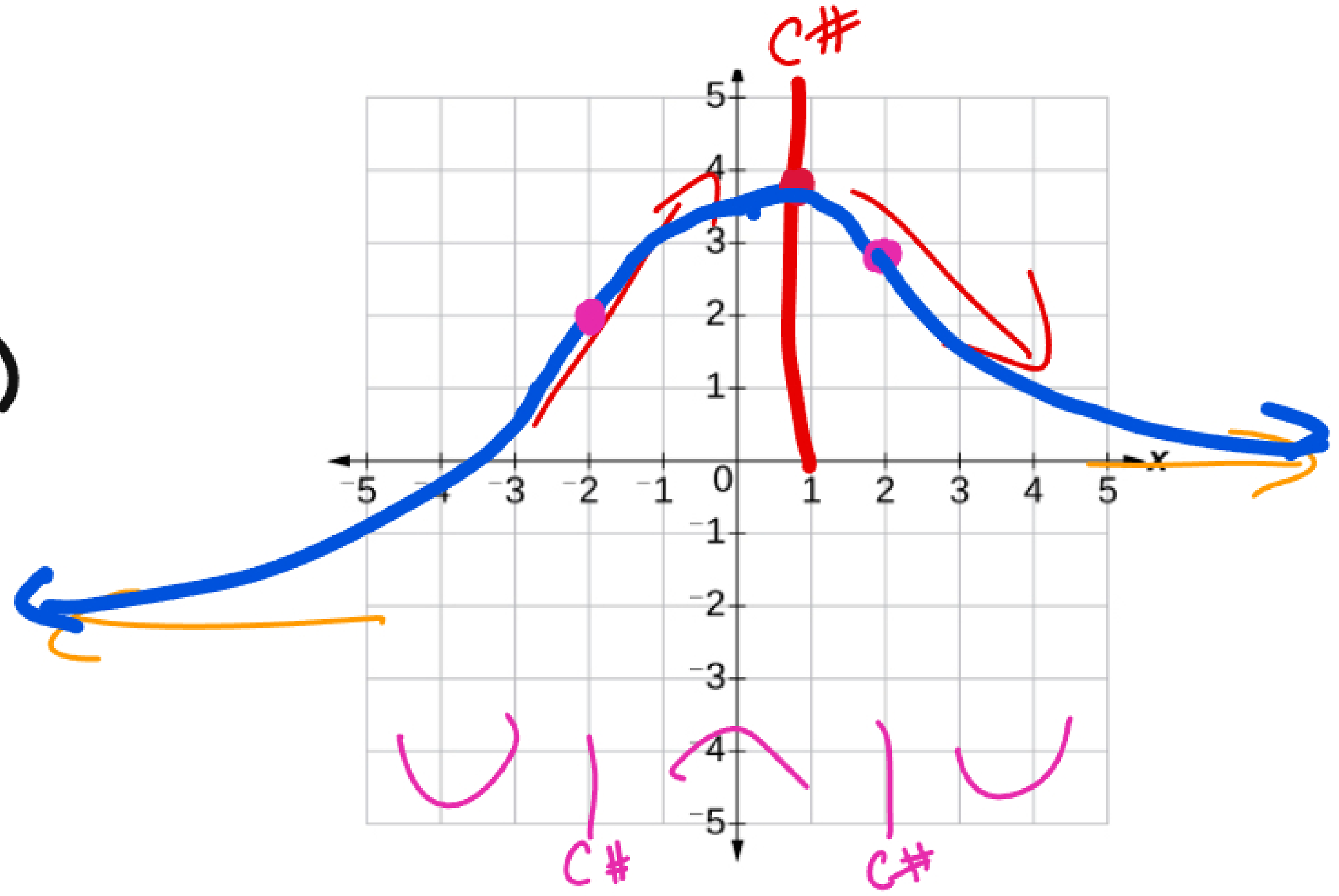
(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

$f'(x) \to (1, 0)$

$f''(x) \to (-2, 0), (2, 0)$

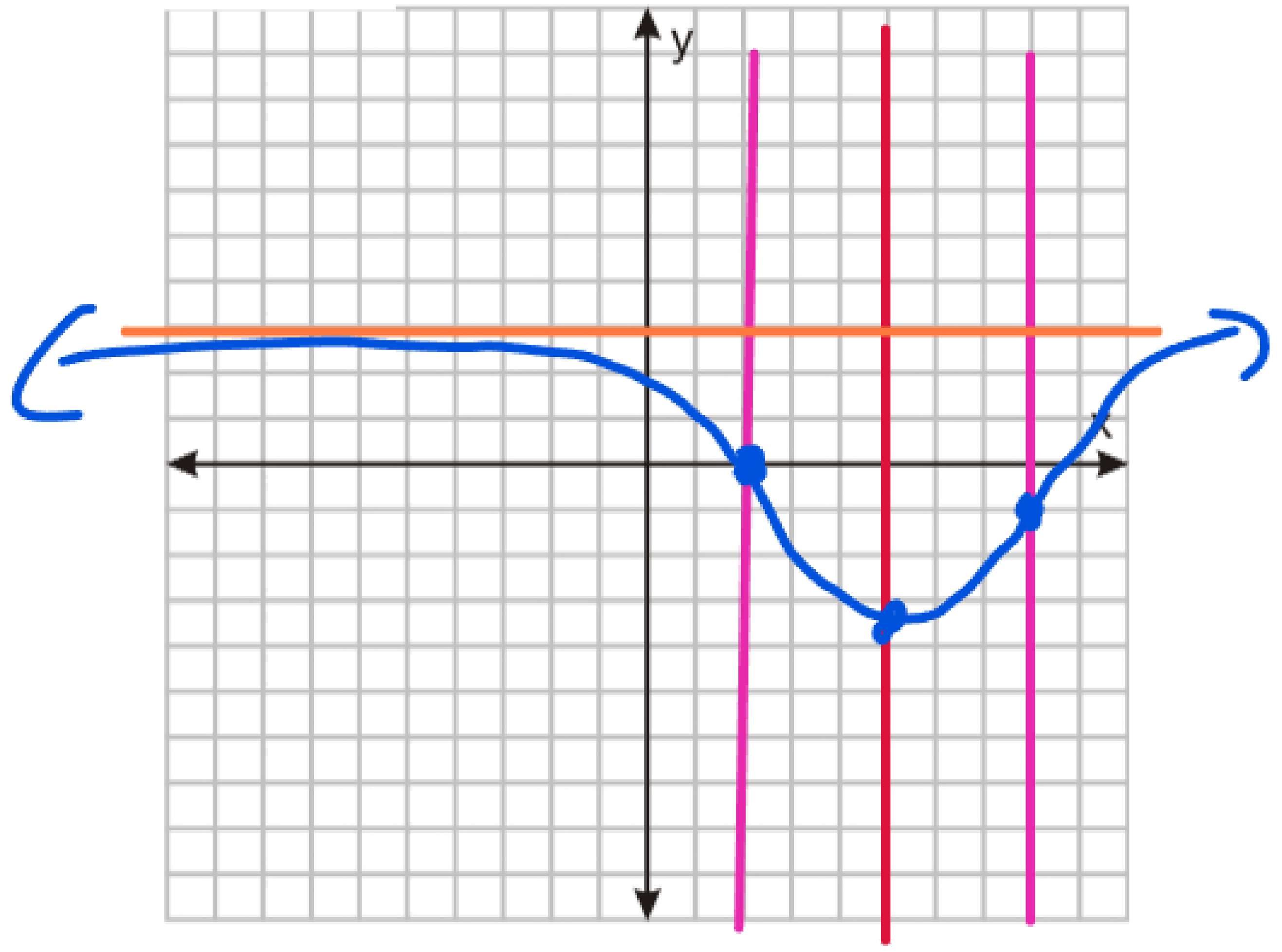


$f'(5) = 0, \quad f'(x) < 0$ when $x < 5$,   5

$f'(x) > 0$ when $x > 5, \quad f''(2) = 0, \quad f''(8) = 0$,   2, 8

$f''(x) < 0$ when $x < 2$ or $x > 8$,

$f''(x) > 0$ for $2 < x < 8, \quad \lim_{x \to \infty} f(x) = 3, \quad \lim_{x \to -\infty} f(x) = 3$ HA 3



$f'(0) = f'(4) = 0$, $\quad f'(x) = 1$ if $x < -1$,

$x = 0, 4, 2$ Asy

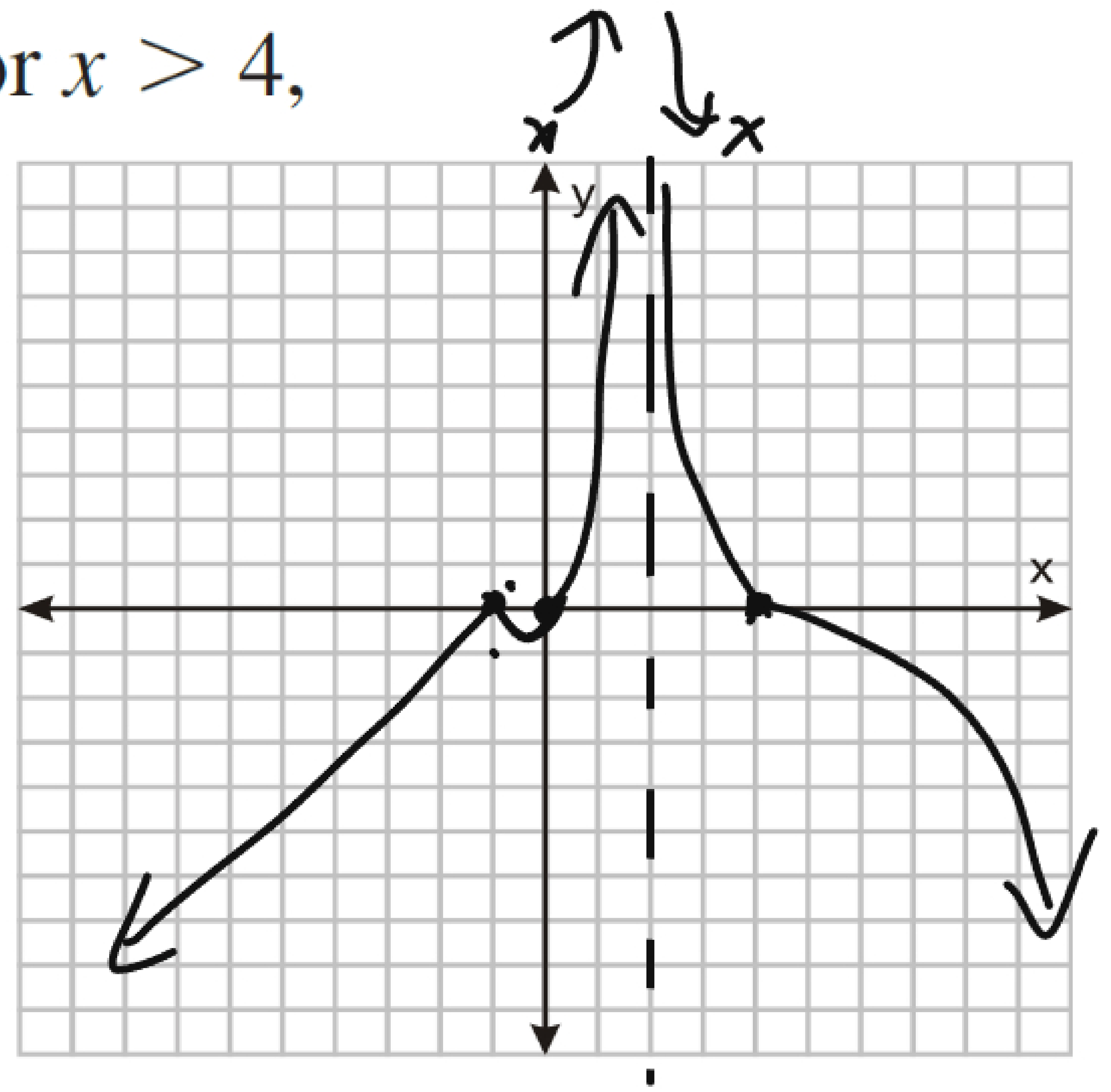
$f'(x) > 0$ if $0 < x < 2$,

$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4$,

$\lim_{x \to 2^-} f'(x) = \infty$, $\quad \lim_{x \to 2^+} f'(x) = -\infty$, ✓

$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4$,

$f''(x) < 0$ if $x > 4$



rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?