Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Indeterminate Forms (Types $\frac{0}{0}, \frac{\infty}{\infty}$)

$$\lim_{x\to 1}\frac{x^2-x}{x^2-1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1} = \frac{1}{2}$$

$$\lim_{x\to \infty}\frac{x^2-1\ /x^2}{2x^2+1\ /x^2} \to \frac{1-1/x^2}{2+1/x^2} = \frac{1}{2}$$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

EXAMPLE 1 Find $\lim_{x\to 1} \dfrac{\ln x}{x-1}$.

$$f(x) = \ln x$$
$$f'(x) = 1/x$$

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} \to \frac{\ln(1)}{1-1} = \frac{0}{0}$$

$$g(x) = x - 1$$
$$g'(x) = 1$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} \overset{H}{=} \lim_{x\to 1} \frac{1/x}{1} = \boxed{1}$$

EXAMPLE 2 Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$$

H

$$\lim_{x \to \infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

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$$\lim_{x \to \infty} \frac{e^x}{2} = \frac{\infty}{2} = \boxed{\infty}$$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$g(x) = x^2$

$g'(x) = 2x$

$g''(x) = 2$

EXAMPLE 3 Calculate $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$. $= \frac{\infty}{\infty}$

$f(x) = \ln x$

$f'(x) = 1/x$

$g(x) = \sqrt{x} = x^{1/2}$

$g'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f''(x) = -1x^{-2} = \frac{-1}{x^2}$

$g''(x) = \frac{-1}{4} x^{-3/2}$

$$\lim_{x \to \infty} \frac{1/x}{1/2\sqrt{x}} = \frac{0}{0}$$

$$\left(\frac{1}{x}\right) \cdot \left(\frac{2\sqrt{x}}{1}\right) = \frac{2\sqrt{x}}{x} = \frac{2x^{1/2}}{x} = \frac{2}{x^{1/2}}$$

$$\lim_{x \to \infty} \frac{2}{\sqrt{x}} \to 0$$

EXAMPLE 4 Find $\lim_{x\to 0} \frac{\tan x - x}{x^3} = \frac{0}{0}$

$f(x) = \tan x - x$

$f'(x) = \sec^2 x - 1$

$f''(x) = 2\sec^2 x \tan x$

$g(x) = x^3$

$g'(x) = 3x^2$

$g''(x) = 6x$

$\overset{H}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \frac{0}{0}$

$\overset{H}{=} \lim_{x\to 0} \frac{\overset{2}{\boxed{2\sec^2 x}}\tan x}{6x}$

$\frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$

$\frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \boxed{\frac{1}{3}}$

$\tan x \to \sec^2 x$

$x \to 1$

$$= 2\sec^2 x \tan x$$

$$\lim_{x \to 0} \frac{3x}{\sin x} \qquad \frac{\lim_{x \to 0} 3 \cdot \lim_{x \to 0} x}{\lim_{x \to 0} \sin x}$$

EXAMPLE 5 Find $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$. $= \frac{0}{2} = 0$ ✓

$$\frac{0}{1-(-1)}$$

Indeterminate Products (Type $0 \cdot \infty$)

This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

EXAMPLE 6 Evaluate $\lim_{x\to 0^+} x \ln x$.

$f(x) = x$

$g(x) = \ln x$

$\ln x \to 1/x$

$1/x \to -1/x^2$

$$\lim_{x\to 0^+} \frac{\ln x}{1/x}$$

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$\frac{1}{x} \div \frac{-1}{x^2}$$

$$\frac{1}{x}\left(\frac{-x^2}{1}\right) = \frac{-x^2}{x} = -x$$

$$\lim_{x\to 0^+} -x \to 0$$

Indeterminate Differences (Type $\infty - \infty$)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

$$\frac{(x-1)-\ln x}{(\ln x)(x-1)}$$

$$\downarrow$$

$$\frac{x-1}{(\ln x)(x-1)} - \frac{\ln x}{\ln(x)(x-1)}$$

$$\lim_{x\to 1^+} \frac{x-1-\ln x}{x\ln x - \ln x} \overset{H}{=} \lim_{x\to 1^+} \frac{1-1/x}{1+\ln x - 1/x}\,\frac{(x)}{(x)}$$

$$= \lim_{x\to 1^+} \frac{x-1}{x + x\ln x - 1} \overset{H}{=} \lim_{x\to 1^+} \frac{1}{1+(1+\ln x)} \to \boxed{\frac{1}{2}}$$

EXAMPLE 8 Calculate $\lim_{x \to \infty} (e^x - x)$.

$$\lim_{x \to \infty} x\left(\frac{e^x}{x} - 1\right)$$

$$\left[\lim_{x \to \infty} (x)\right]\left[\lim_{x \to \infty} \frac{e^x}{x} - 1\right]$$

$$(\infty)(\infty) = \boxed{\infty}$$

Indeterminate Powers (Types 0^0, ∞^0, 1^∞)

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

$$\ln y = \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = \frac{\frac{4}{1}}{1} = 4$$

$$\lim_{x\to 0^+} \ln y = 4 \rightarrow y = \boxed{e^4}$$

$$\ln(1+\sin 4x)$$
$$\downarrow$$
$$\frac{4\cos 4x}{1+\sin 4x}$$

$$\tan x \rightarrow \sec^2 x$$

EXAMPLE 10 Find $\lim_{x \to 0^+} x^x$.