**A. Domain** It's often useful to start by determining the domain $D$ of $f$, that is, the set of values of $x$ for which $f(x)$ is defined.

1) $\dfrac{1}{x}$    $x \neq 0$

2) $\sqrt[even]{x}$    $x \geq 0$

3) $Ln(x)$    $x > 0$

4) Real Life Restriction

**B. Intercepts** The $y$-intercept is $f(0)$ and this tells us where the curve intersects the $y$-axis. To find the $x$-intercepts, we set $y = 0$ and solve for $x$. (You can omit this step if the equation is difficult to solve.)

every function has one y-Intercept

X-Intercepts    y = 0  Solve

## C. Symmetry  $\Rightarrow$ even

(i) If $f(-x) = f(x)$ for all $x$ in $D$, that is, the equation of the curve is unchanged when $x$ is replaced by $-x$, then $f$ is an *even function* and the curve is symmetric about the $y$-axis. (See Section 1.1.) This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the $y$-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$. $\rightarrow$ odd

(ii) If $f(-x) = -f(x)$ for all $x$ in $D$, then $f$ is an *odd function* and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = 1/x$, and $y = \sin x$.

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(iii) If $f(x + p) = f(x)$ for all $x$ in $D$, where $p$ is a positive constant, then $f$ is a **periodic function** and the smallest such number $p$ is called the **period**. For instance, $y = \sin x$ has period $2\pi$ and $y = \tan x$ has period $\pi$. If we know what the graph looks like in an interval of length $p$, then we can use translation to visualize the entire graph (see Figure 4).
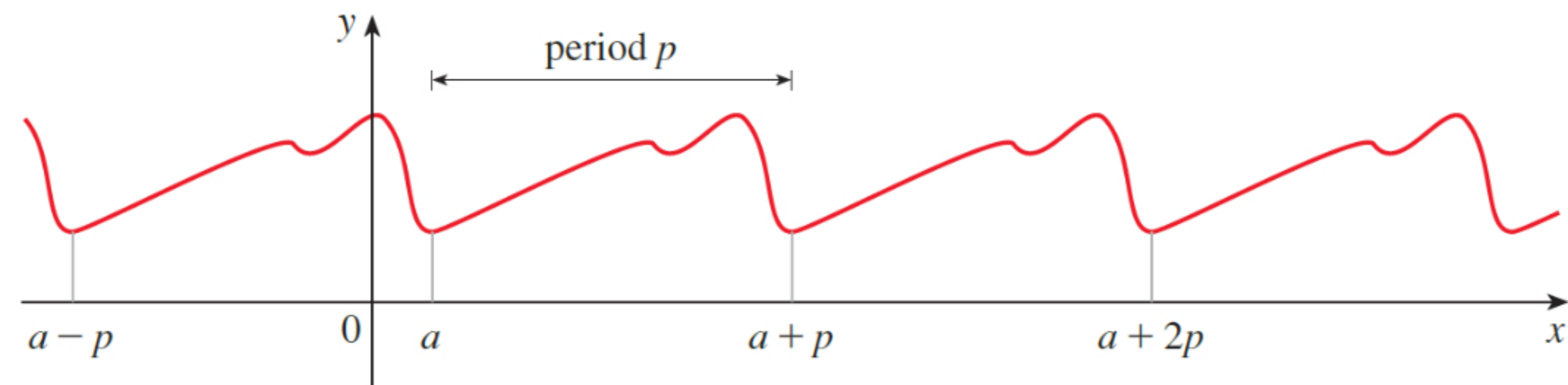
## D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 2.6 that if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x \to a^+} f(x) = \infty \qquad\qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad\qquad \lim_{x \to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but $a$ is an endpoint of the domain of $f$, then you should compute $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

**E. Intervals of Increase or Decrease** Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive ($f$ is increasing) and the intervals on which $f'(x)$ is negative ($f$ is decreasing).

$$(-)\quad D$$
$$(+)\quad I$$
$$(-)\quad D$$
$$(+)\quad I$$

$$-5 \qquad 2 \qquad 5$$
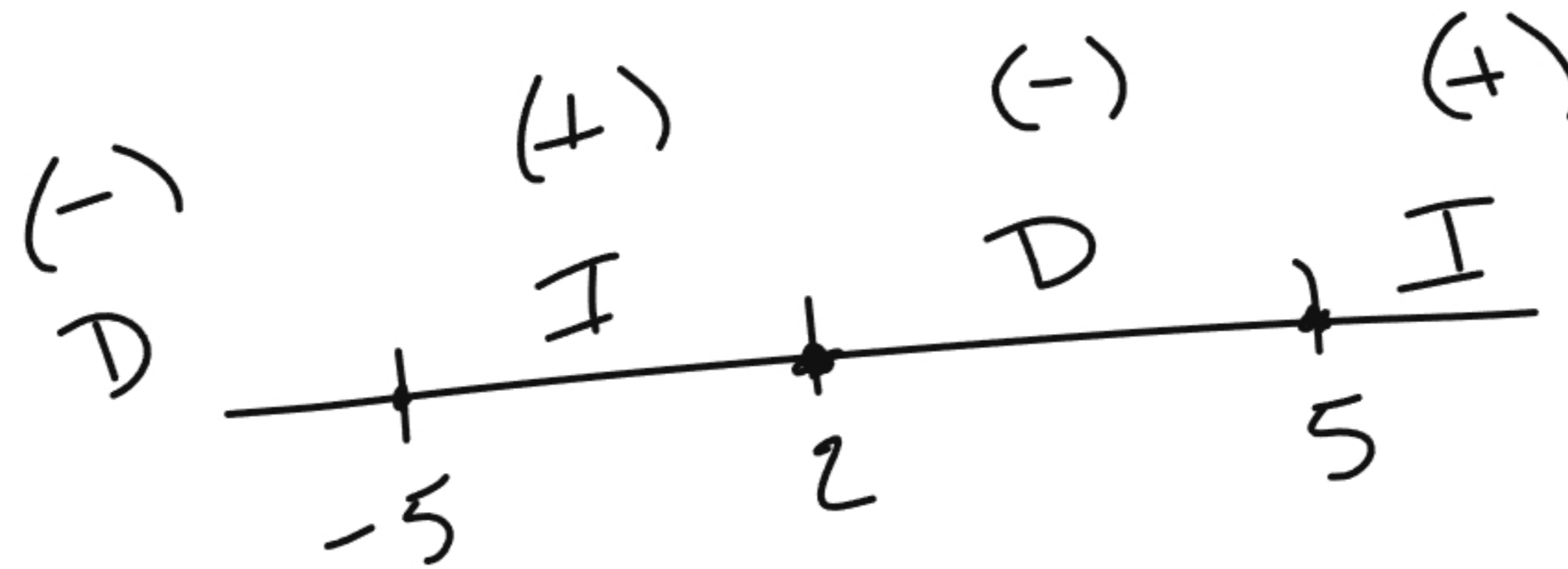
**F. Local Maximum or Minimum Values** Find the critical numbers of $f$ [the numbers $c$ where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If $f'$ changes from positive to negative at a critical number $c$, then $f(c)$ is a local maximum. If $f'$ changes from negative to positive at $c$, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

I    ↙max    D
↓
Concave down
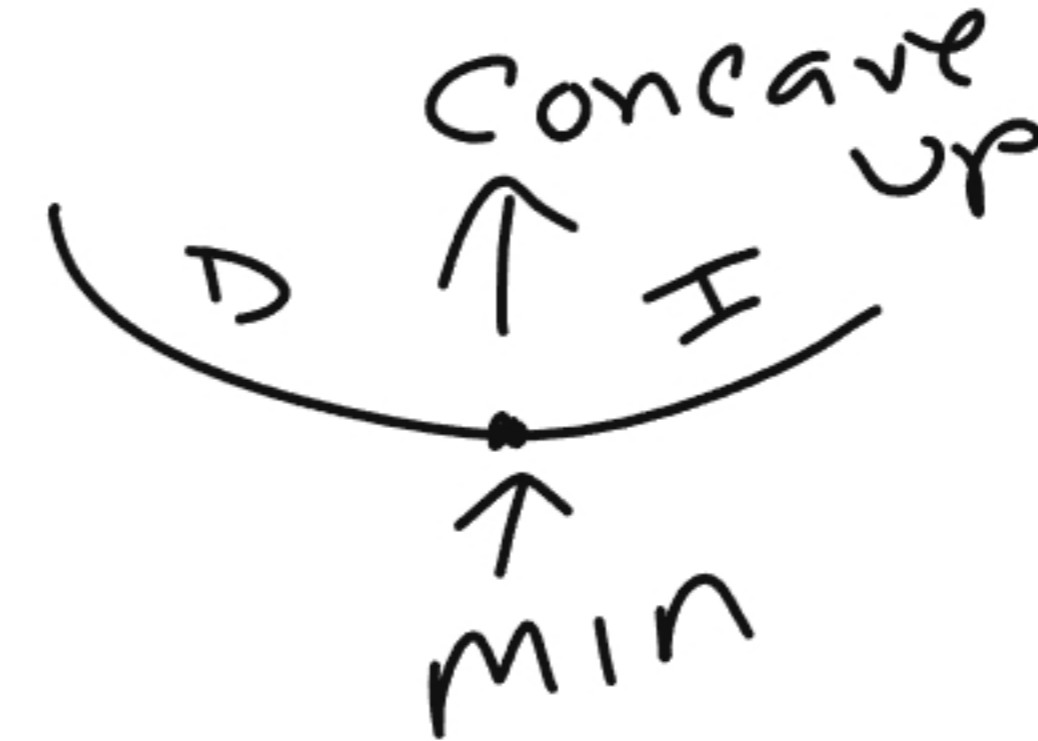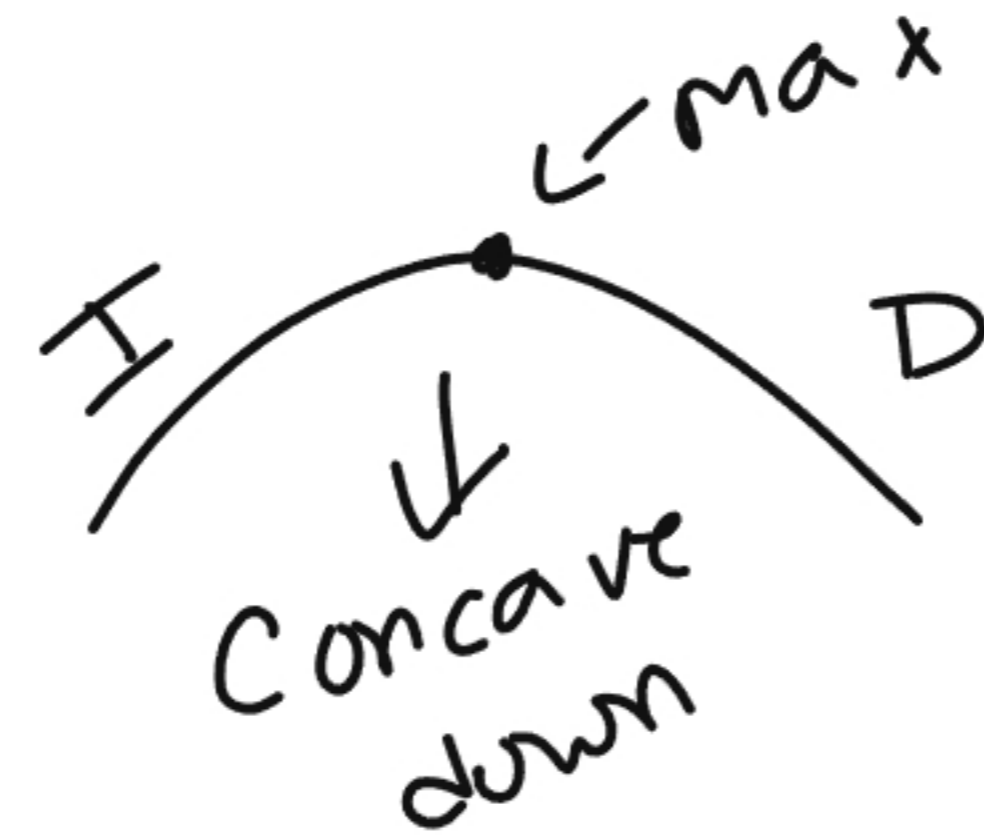
Concave up
D    ↑    I
↑
min

**G. Concavity and Points of Inflection** Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

up    down

↖ Point of Inflection
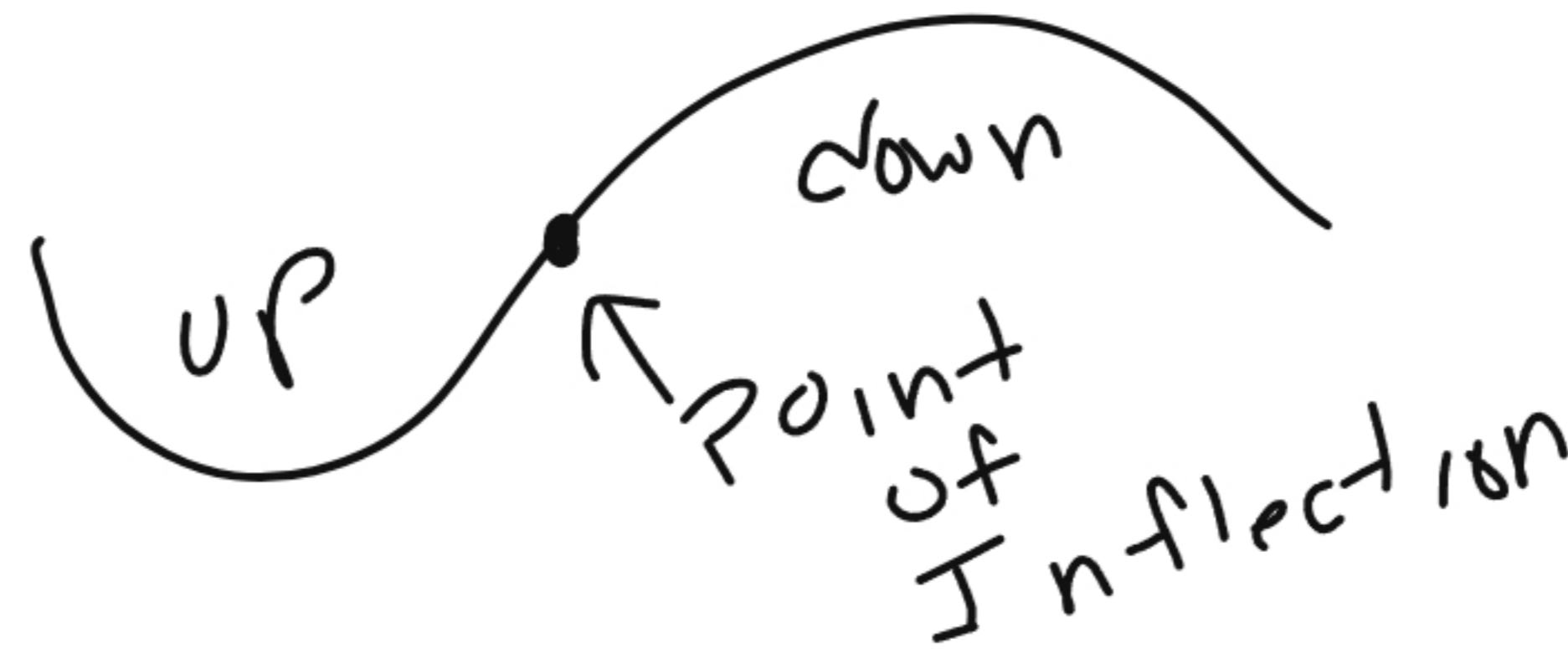
$$y = \frac{2x^2}{x^2 - 1}$$

$D \Rightarrow x^2 - 1 \neq 0$

$$x^2 \neq 1$$

$$x \neq \sqrt{1} = \pm 1$$

$D \rightarrow (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
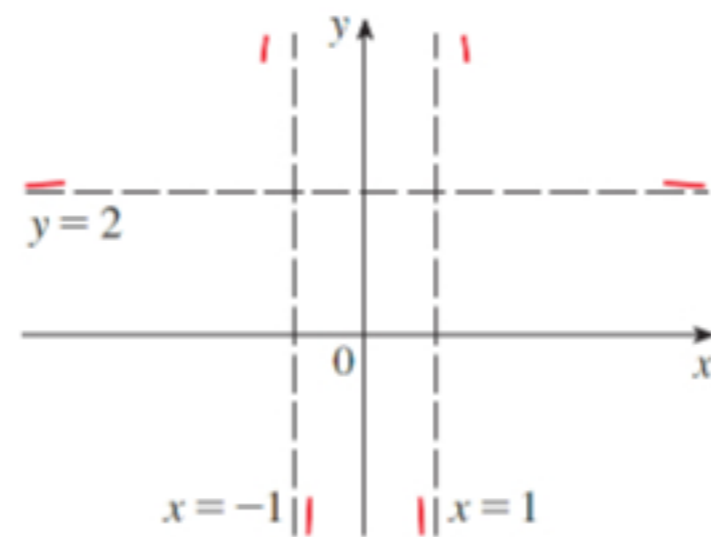
$$y = \frac{2x^2}{x^2 - 1}$$

y-Intercept $\rightarrow$ (0, 0)

$f(0) = \frac{2(0)^2}{(0)^2 - 1} = \frac{0}{-1} = 0$

X-Intercept $\rightarrow$ (0, 0)

$0 = \frac{2x^2}{x^2 - 1} \rightarrow 2x^2 = 0$

$x^2 = 0$

$x = 0$

$$y = \frac{2x^2}{x^2 - 1}.$$

$D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Vertical Asymptotes $x = -1$

H.A.

$\lim\limits_{x \to \infty} \dfrac{2x^2/x^2}{x^2 - 1 \,/x^2}$

$\boxed{y = 2}$

$\lim\limits_{x \to -1} \dfrac{2x^2}{x^2 - 1}$

$\nearrow \lim\limits_{x \to -1^+} \dfrac{2x^2}{x^2 - 1} = \infty$

$\searrow \lim\limits_{x \to -1^-} \dfrac{2x^2}{x^2 - 1} = -\infty$



$y = 2$

$x = -1$   $x = 1$

$\lim\limits_{x \to 1} \dfrac{2x^2}{x^2 - 1}$

$\nearrow x \to 1^+ \quad \infty$

$\searrow x \to 1^- \quad \infty$

$$y = \frac{2x^2}{x^2 - 1}$$

$$\frac{f'g - g'f}{g^2}$$

$$\frac{n'd - d'n}{d^2}$$
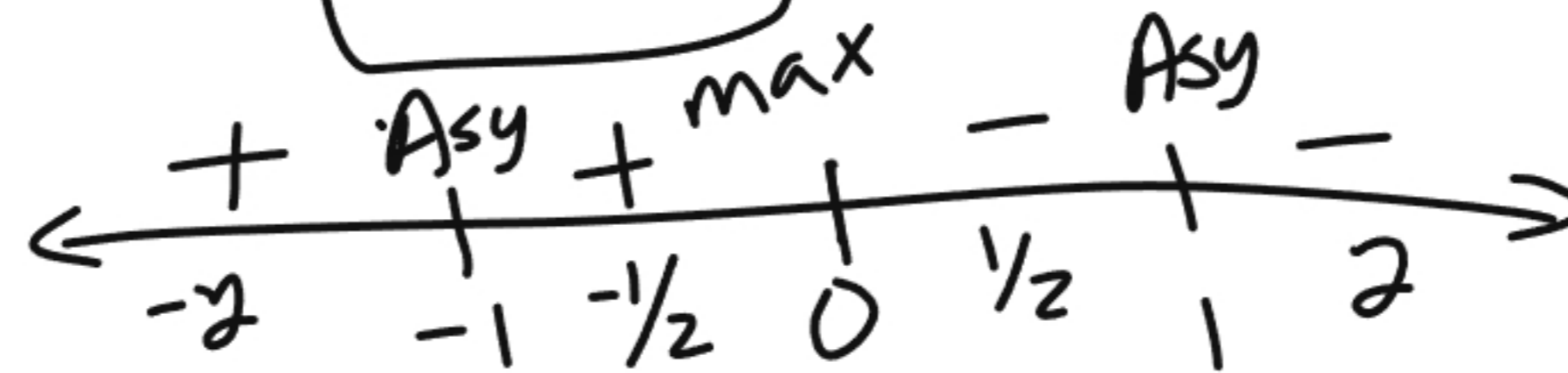
$$y' = \frac{-4x}{(x^2-1)^2} = 0$$

$$-4x = 0$$

$$\boxed{\begin{array}{l} x = 0 \\ x = 1 \\ x = -1 \end{array}}$$ Critical #'s

$$x^2 - 1 \neq 0$$

$$x \neq \pm 1$$

$$f(x)$$
$$I \to (-\infty, -1) \cup (-1, 0)$$
$$D \to (0, 1) \cup (1, \infty)$$

Local max $(0, 0)$

$$y = \frac{2x^2}{x^2 - 1}$$

$$y' = \frac{-4x}{(x^2-1)^2}$$

$$y'' = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

$-4x \rightarrow -4$

$(x^2-1)^2 \rightarrow (2x)(2)(x^2-1)$

$$= 4x(x^2 - 1)$$

$$\frac{-4(x^2-1)^2 - (4x(x^2-1))(-4x)}{(x^2-1)^4}$$

$$\frac{-4(x^2-1)^2 + 16x^2(x^2-1)}{(x^2-1)^4}$$

$$\frac{-4(x^2-1) + 16x^2}{(x^2-1)^3} = \frac{12x^2 + 4}{(x^2-1)^3}$$

$\textcolor{red}{-4x^2 + 4}$

$$y = \frac{2x^2}{x^2 - 1}$$

$$y' = \frac{-4x}{(x^2-1)^2}$$

$$y'' = \frac{12x^2+4}{(x^2-1)^3} > 0$$
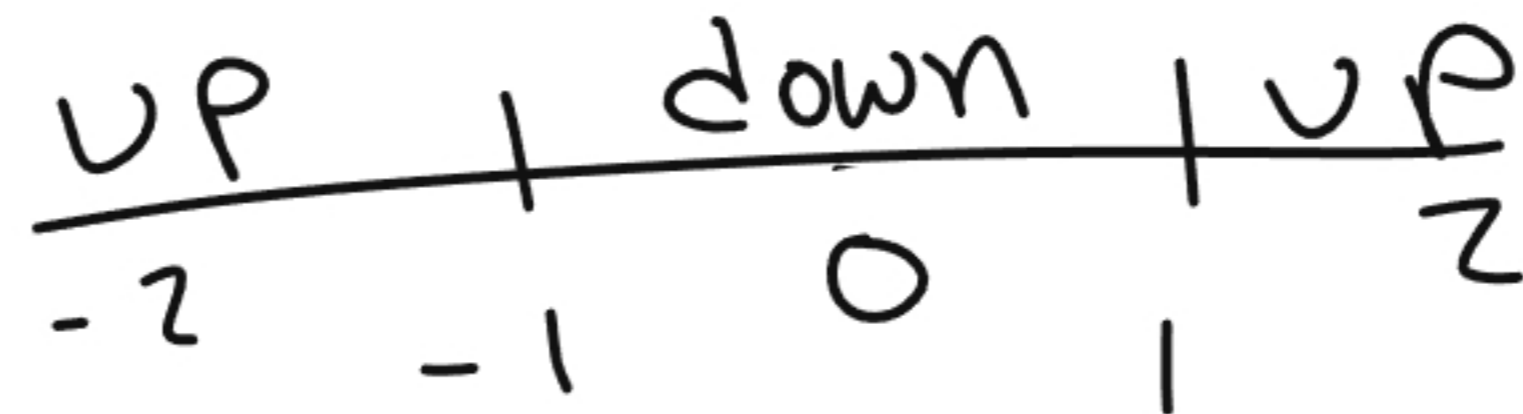
Concave up
$(-\infty, -1) \cup (1, \infty)$

Concave down
$(-1, 1)$

$$12x^2 + 4 = 0$$

$$12x^2 = -4$$

$$\sqrt{x^2} = \sqrt{\frac{-4}{12}}$$

$$x^2 - 1 \neq 0 \qquad x = \pm 1$$

up | down | up

-2    -1    0    1    2

$$f(x) = \frac{\cos x}{2 + \sin x}.$$

$2 + \sin x \neq 0$

$\sin x \neq -2$   $D \Rightarrow \mathbb{R}$

period $\Rightarrow 2\pi$

$0 \rightarrow 2\pi$

Intercepts

y-Intercept   $\frac{\cos(0)}{2 + \sin(0)} = \frac{1}{2}$   $(0, \frac{1}{2})$

x-Intercept   $\frac{\cos(x)}{2 + \sin(x)} = 0 \rightarrow \cos(x) = 0$

$X = \frac{\pi}{2}, \frac{3\pi}{2}$

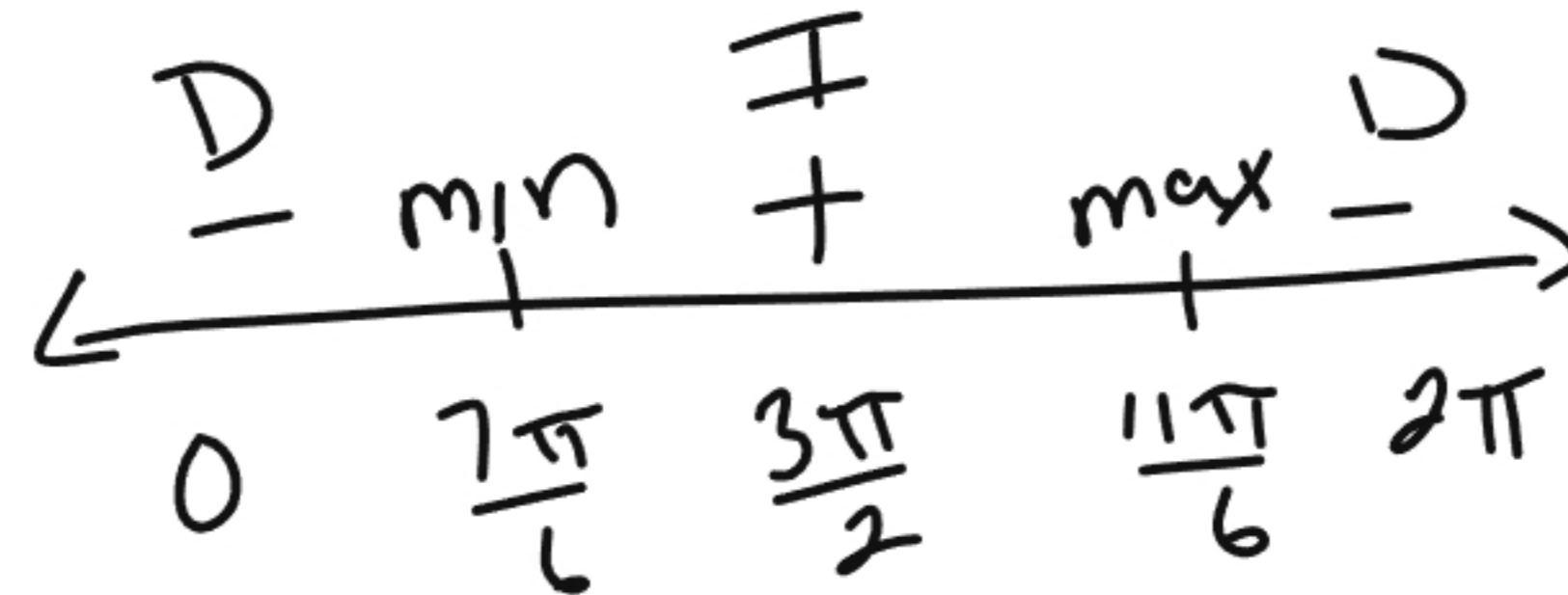$X = \frac{\pi}{2} + k\pi$

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$$f(x) = \frac{\cos x}{2 + \sin x}.$$

$$-2\sin x - \sin^2 x - \cos^2(x)$$
$$\rightarrow -(2\sin x + \sin^2 x + \cos^2 x)$$

$$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x \, (\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

$$-2\sin x - 1 = 0$$

$$\sin x = \frac{-1}{2}$$

$$X = \frac{11\pi}{6}, \frac{7\pi}{6}$$

D        I
—   min  +   max   D
←———+————+————+————→
0    $\frac{7\pi}{6}$   $\frac{3\pi}{2}$   $\frac{11\pi}{6}$   $2\pi$

$$f(x) = \frac{\cos x}{2 + \sin x}.$$

$\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$ $\frac{7\pi}{6}$

$\frac{11\pi}{6}$ $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

$$X = \frac{7\pi}{6} \rightarrow \frac{\cos \frac{7\pi}{6}}{2 + \sin\left(\frac{7\pi}{6}\right)} = \frac{-\sqrt{3}/2}{2 + \left(-\frac{1}{2}\right)} = \frac{\frac{-\sqrt{3}}{2}}{3/2} = \boxed{\frac{-\sqrt{3}}{3}}$$

$$X = \frac{11\pi}{6} \rightarrow \frac{\cos\left(\frac{11\pi}{6}\right)}{2 + \sin\left(\frac{11\pi}{6}\right)} = \frac{\frac{\sqrt{3}}{2}}{2 + \left(-\frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{2}}{3/2} = \boxed{\frac{\sqrt{3}}{3}}$$

$$\min\left(\frac{7\pi}{6}, \frac{-\sqrt{3}}{3}\right) \qquad \max\left(\frac{11\pi}{6}, \sqrt{3}/3\right)$$