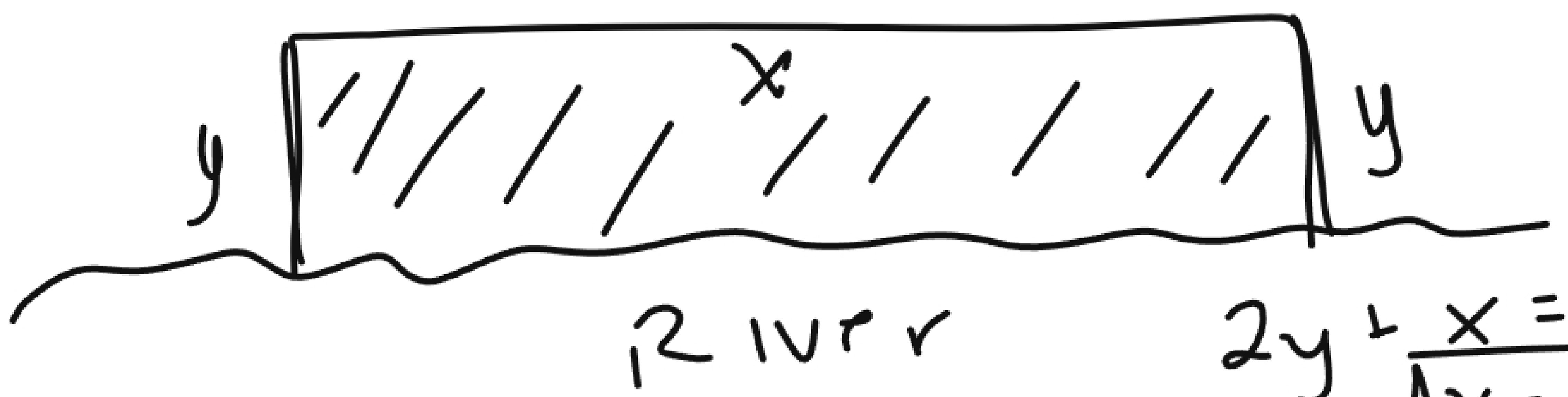
Steps In Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols ($a, b, c, \ldots, x, y$) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.
6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

fence 2400ft



$2y + x = 2400$

$\boxed{x = 2400 - 2y}$

$A = LW$

$A = xy$

$A = (2400 - 2y)(y)$

BitPaper
Share Knowledge

$A = (2400 - 2y)(y)$

$A = 2400y - 2y^2$

$A' = 2400 - 4y = 0$

$A'' = -4$

$2400 - 4y = 0$

$-4y = -2400$

$y = 600$

max

$y = 600$

$x = 2400 - 2y$

$x = 1200$

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

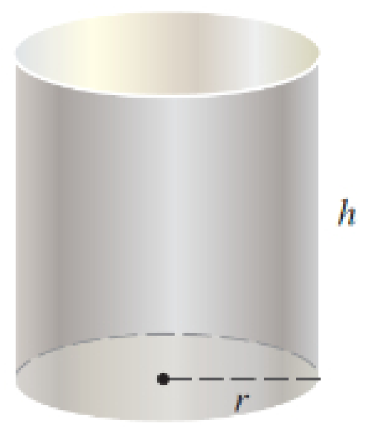
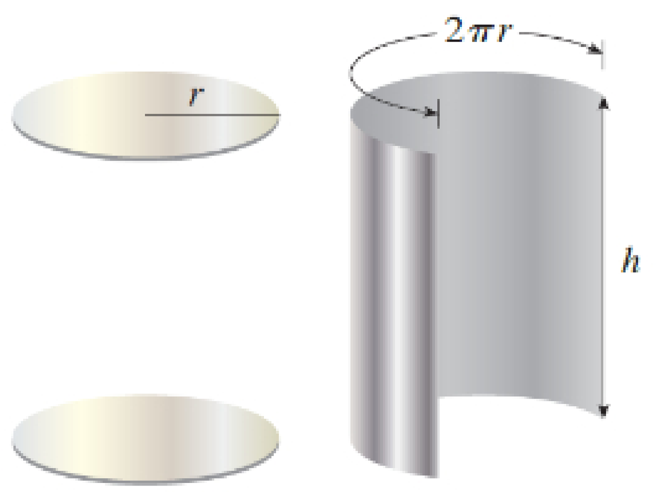


FIGURE 3



Area $2(\pi r^2)$ Area $(2\pi r)h$

FIGURE 4

$$SA = 2\pi r h + 2(\pi r^2)$$

$$V = \pi r^2 h = 1000 \text{ mL}$$

$$h = \frac{1000}{\pi r^2}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$SA = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$$

BitPaper
Share Knowledge

$$SA = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$$

$$SA = \frac{2000}{r} + 2\pi r^2$$

$$\frac{2000}{r} = 2000r^{-1}$$

$$-2000r^{-2}$$

$$SA' = \frac{-2000}{r^2} + \frac{4\pi r^3}{r^2} = 0$$

$$= \frac{-2000 + 4\pi r^3}{r^2}$$

$$-2000 + 4\pi r^3 = 0$$

$$4\pi r^3 = 2000$$

$$\pi r^3 = 500$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{500/\pi}$$

BitPaper
Share Knowledge

$$\boxed{r = \sqrt[3]{\frac{500}{\pi}}}$$

$$h = \frac{1000}{\pi r^2}$$

$$\boxed{h = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{2/3}}}$$

$\sqrt[3]{\frac{500}{\pi}}$	$r \quad = 5.419260701$
$\frac{1000}{\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}}$	$h \quad = 10.8385214$

EXAMPLE 4 A woman launches her boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). She could row her boat directly across the river to point C and then run to B, or she could row directly to B, or she could row to some point D between C and B and then run to B. If she can row 6 km/h and run 8 km/h, where should she land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the woman rows.)

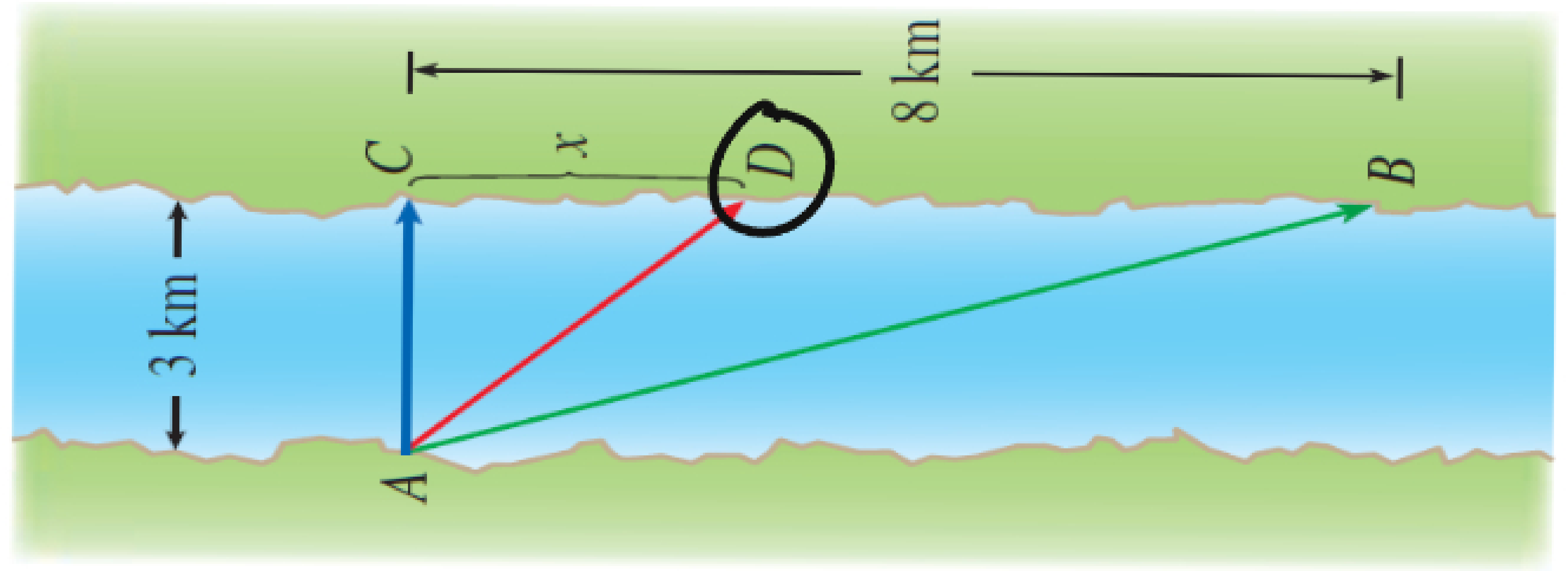


FIGURE 7

$$\overline{AD} = \frac{\sqrt{3^2 + x^2}}{6} \quad \text{Boat}$$

$$+ \qquad\qquad + $$

$$\overline{DB} = \frac{8 - x}{8} \quad \text{Run}$$

$$\text{Time} = \boxed{\frac{\sqrt{9+x^2}}{6}} + \boxed{\frac{8-x}{8}}$$

$$\frac{\sqrt{9+x^2}}{6} = \frac{1}{6}\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{9+x^2}}\right)$$

$$T' = \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8}$$

$$\sqrt{u} = \frac{1}{2\sqrt{u}}$$

$$0 = \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8}$$

$$\frac{1}{8} = \frac{x}{6\sqrt{9+x^2}}$$

$$6\sqrt{9+x^2} = 8x$$

$$36(9+x^2) = 64x^2$$

$$324 + 36x^2 = 64x^2$$

$$324 = 28x^2$$

$$\sqrt{\frac{324}{28}} = \sqrt{x^2} \to \boxed{x = \frac{9}{\sqrt{7}}}$$

$$\frac{81}{7}$$

$$\frac{8-x}{8} \to \frac{1}{8}(-1)$$

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

EXAMPLE 6 A store has been selling 200 TV monitors a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of monitors sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize revenue? $X = 12.5$

R today $= 200(350) = 70{,}000$

$R = \boxed{(350 - 10x)}\boxed{(200 + 20x)}$

\$225

450 TV/wk

$R = 70000 + 5000x - 200x^2$

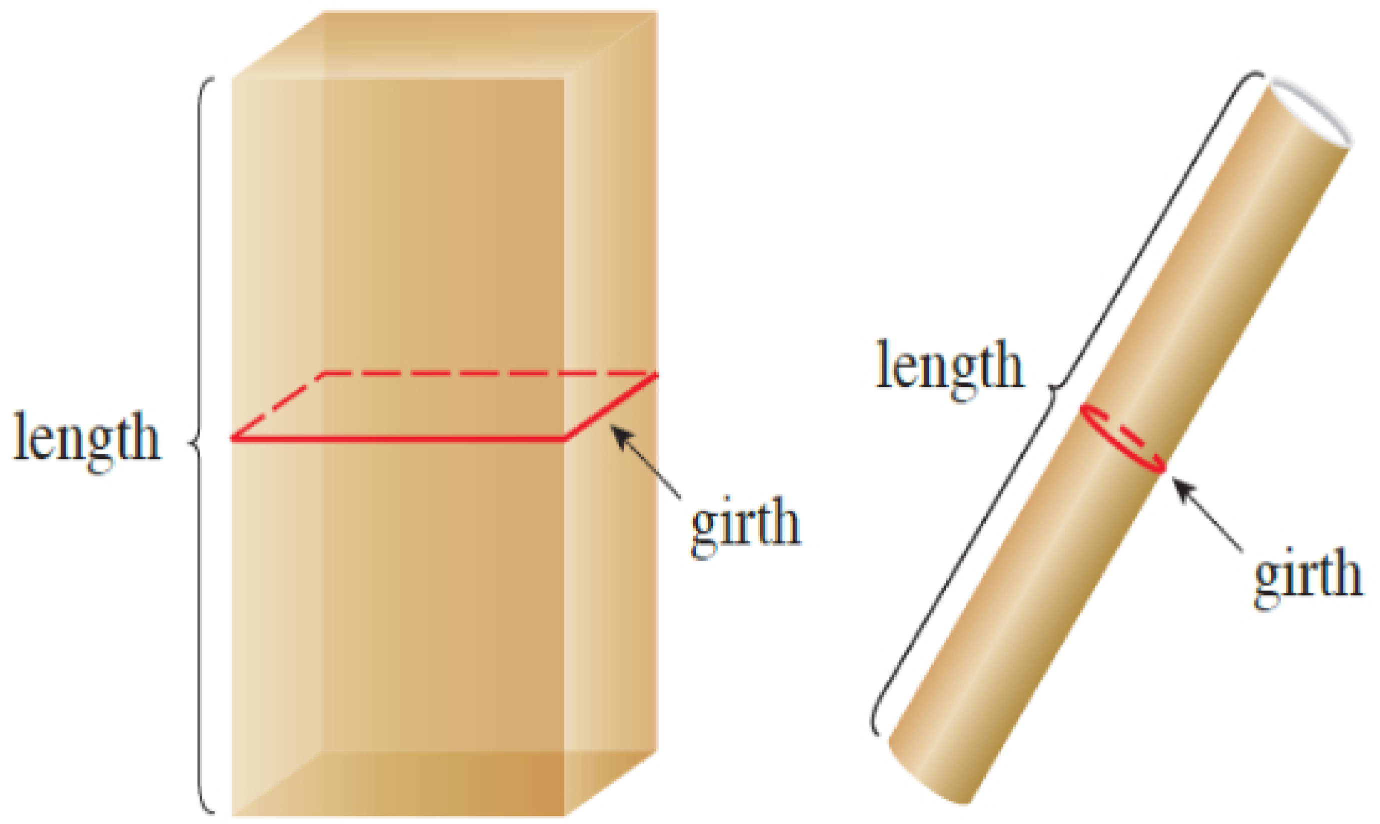
$R' = 5000 - 400x = 0$

$5000 = 400x$

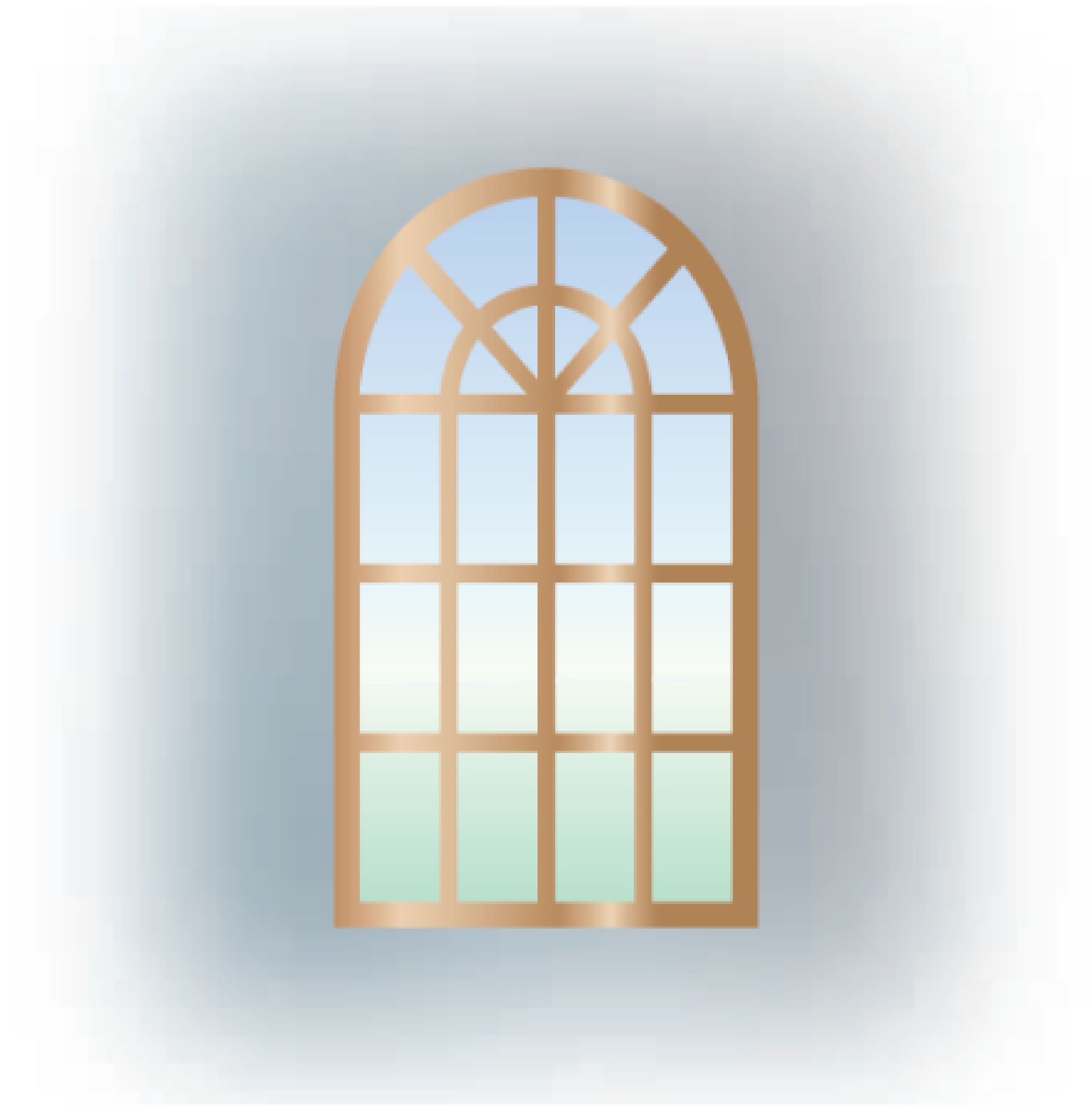
$12.5 = X$

$R'' = -400$

23. A package to be mailed using the US postal service may not measure more than 108 inches in length plus girth. (Length is the longest dimension and girth is the largest distance around the package, perpendicular to the length.) Find the dimensions of the rectangular box with square base of greatest volume that may be mailed.



A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 1.1.72.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.



57. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?