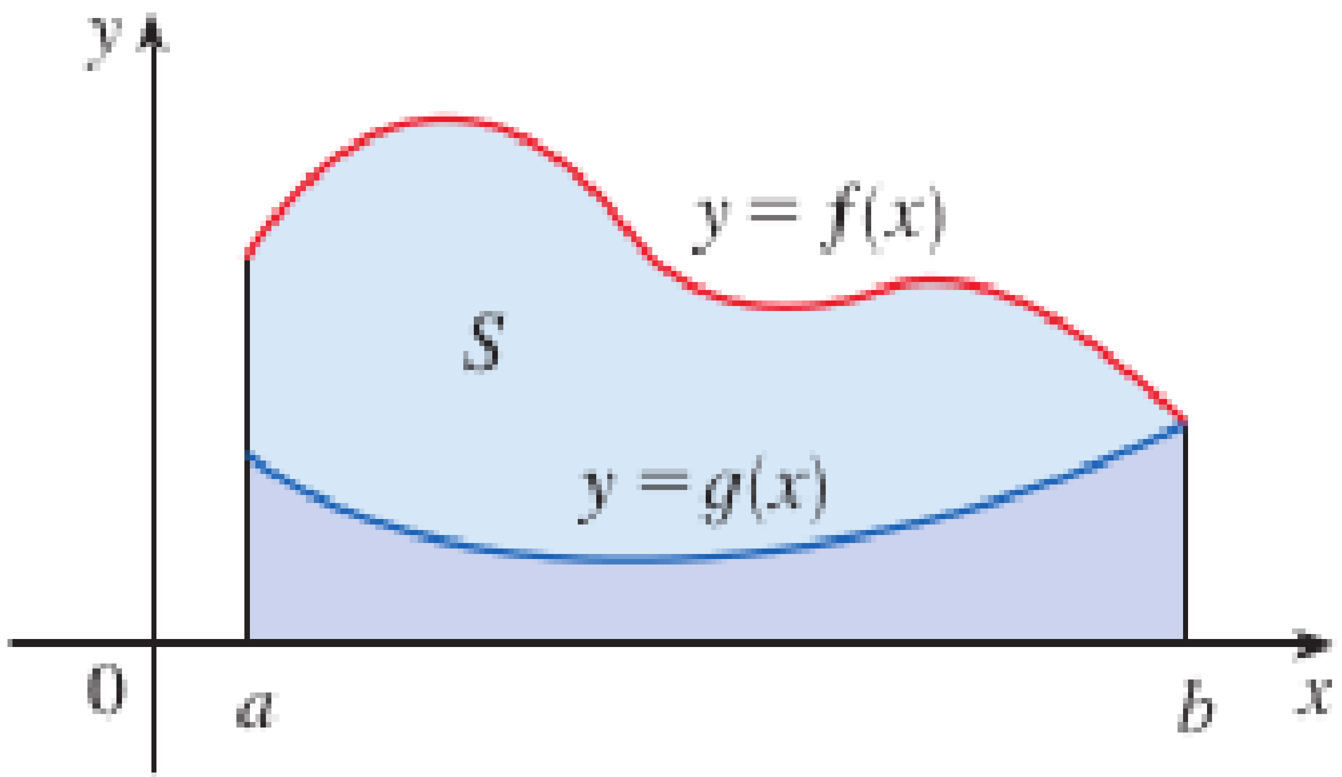
6.1. Areas between Curves



Details

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

$$f(x) = e^x \qquad g(x) = x$$

$$\int_0^1 e^x \, dx - \int_0^1 x \, dx$$

$$e^x \Big|_0^1 - \frac{x^2}{2} \Big|_0^1 = \left(e - \frac{1}{2}\right) - (1 - 0)$$

$$e - 3/2$$

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

$$x^2 = 2x - x^2$$

$$2x^2 - 2x = 0$$

$$2x(x-1) = 0$$

$$2x = 0 \qquad x - 1 = 0$$

$$x = 0 \qquad x = 1$$

$$\int_0^1 (2x - x^2) - (x^2)\,dx$$

$$\int_0^1 2x - 2x^2\,dx$$

$$x^2 - \frac{2x^3}{3}\Big|_0^1$$

$$\left(1 - \frac{2}{3}\right) = \boxed{\frac{1}{3}}$$

Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2 + 1}$ and $y = x^4 - x$.

$$y = \frac{x}{\sqrt{x^2+1}} \qquad y = x^4 - x$$

$$\int_0^{1.18} \left(\frac{x}{\sqrt{x^2+1}}\right) - (x^4 - x)\,dx$$

$$\int_0^{1.18} \boxed{\left(\frac{x}{\sqrt{x^2+1}}\right)} - (x^4 - x)\,dx$$

$$\int \frac{\boxed{x}}{\sqrt{\boxed{x^2+1}}}\,\boxed{dx}$$

$$u = \boxed{x^2+1}$$

$$du = 2x\,dx$$

$$\frac{1}{2}\,du = \boxed{x\,dx}$$

$$\int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$\frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) = u^{1/2} = \sqrt{u}$$

$$\sqrt{x^2+1}$$

$$\int_0^{1.18} \left(\frac{x}{\sqrt{x^2+1}}\right) - (x^4 - x)\, dx$$

$$-x^4 + x$$

$$\sqrt{x^2+1} - \frac{x^5}{5} + \frac{x^2}{2} \Bigg|_0^{1.18} = 1.785 - 1$$

$$= \boxed{0.785}$$

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

$y = \sin x \qquad y = \cos x \qquad 0 \le x \le \frac{\pi}{2}$

$$\int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$(\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$\sin x$

$\cos x$

$-\sin x$

$-\cos x$

$\frac{d}{dx} \downarrow \qquad \uparrow \int$

$\int \cos x = \sin x$

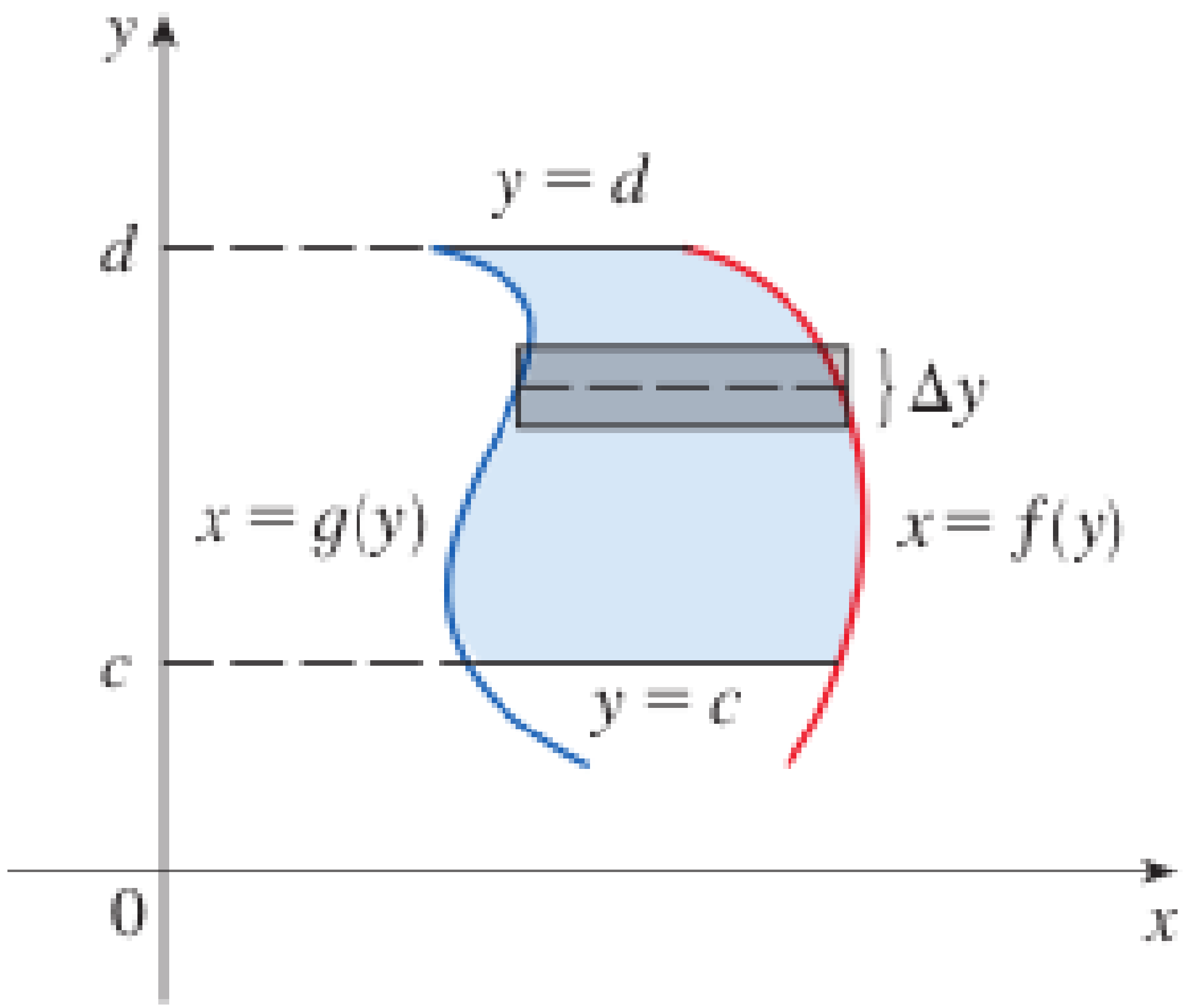
$\int \sin x = -\cos x$

$$\left(\sin x + \cos x\right)\Big|_0^{\pi/4} + \left(-\cos x - \sin x\right)\Big|_{\pi/4}^{\pi/2}$$

$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) + (0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$\sqrt{2} - 1 - 1 + \sqrt{2} = \boxed{2\sqrt{2} - 2}$$

$$A = \int_{c}^{d} [f(y) - g(y)]\, dy$$



Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

$y = x - 1$

$y + 1 = x$

$y^2 = 2x + 6$

$\frac{1}{2}y^2 - 3 = x$

$$\int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$$

$$\int_{-2}^{4} (y+1) - (1/2\, y^2 - 3)\, dy$$

$+4$

$$\frac{y^2}{2} - \frac{y^3}{6} + 4y \Big|_{-2}^{4}$$

$$\left(8 - \frac{64}{6} + 16\right) - \left(2 + \frac{8}{6} - 8\right)$$

$$30 - \frac{72}{6} = \boxed{18}$$

58. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leqslant t \leqslant 10$.

What does this area represent?

$$b(t) = 2200e^{0{,}024t}$$

$$d(t) = 1460e^{0{,}018t} \qquad 0 \le t \le 10$$

$$\int_0^{10} 2200e^{0.024t} - 1460e^{0{,}018t} \, dt$$

$$\int 2200\, e^{0.024t}\, dt$$

$$u = 0.024t$$

$$du = 0.024\, dt$$

$$\frac{1}{0.024}\, du = dt$$

$$\frac{2200}{0.024} \int e^{u}\, du$$

$$\int \boxed{2200}\, e^{u} \boxed{\left(\frac{1}{0.024}\right)} du$$

$$\frac{275000}{3} e^{0.024t}$$

$$\int 1460 e^{0.018t}$$

$$u = 0.018t$$

$$du = 0.018 dt$$

$$\frac{1}{0.018} du = dt$$

$$\frac{1460}{0.018} \int e^{u} \, du$$

$$\frac{730000}{9} e^{0.018t}$$

$$\int_0^{10} 2200e^{0.024t} - 1460e^{0.018t}\,dt$$

$$\frac{275000}{3}e^{0.024t} - \frac{730000}{9}e^{0.018t}\Big|_0^{10} \quad \boxed{8868}$$

$\frac{275000}{3}e^{0.024x} - \frac{730000}{9}e^{0.018x}$	$= 19423.54155$
$x = 10$	$= 10$
$\frac{275000}{3}e^{0.024x} - \frac{730000}{9}e^{0.018x}$	$= 10555.55556$
$x = 0$	$= 0$