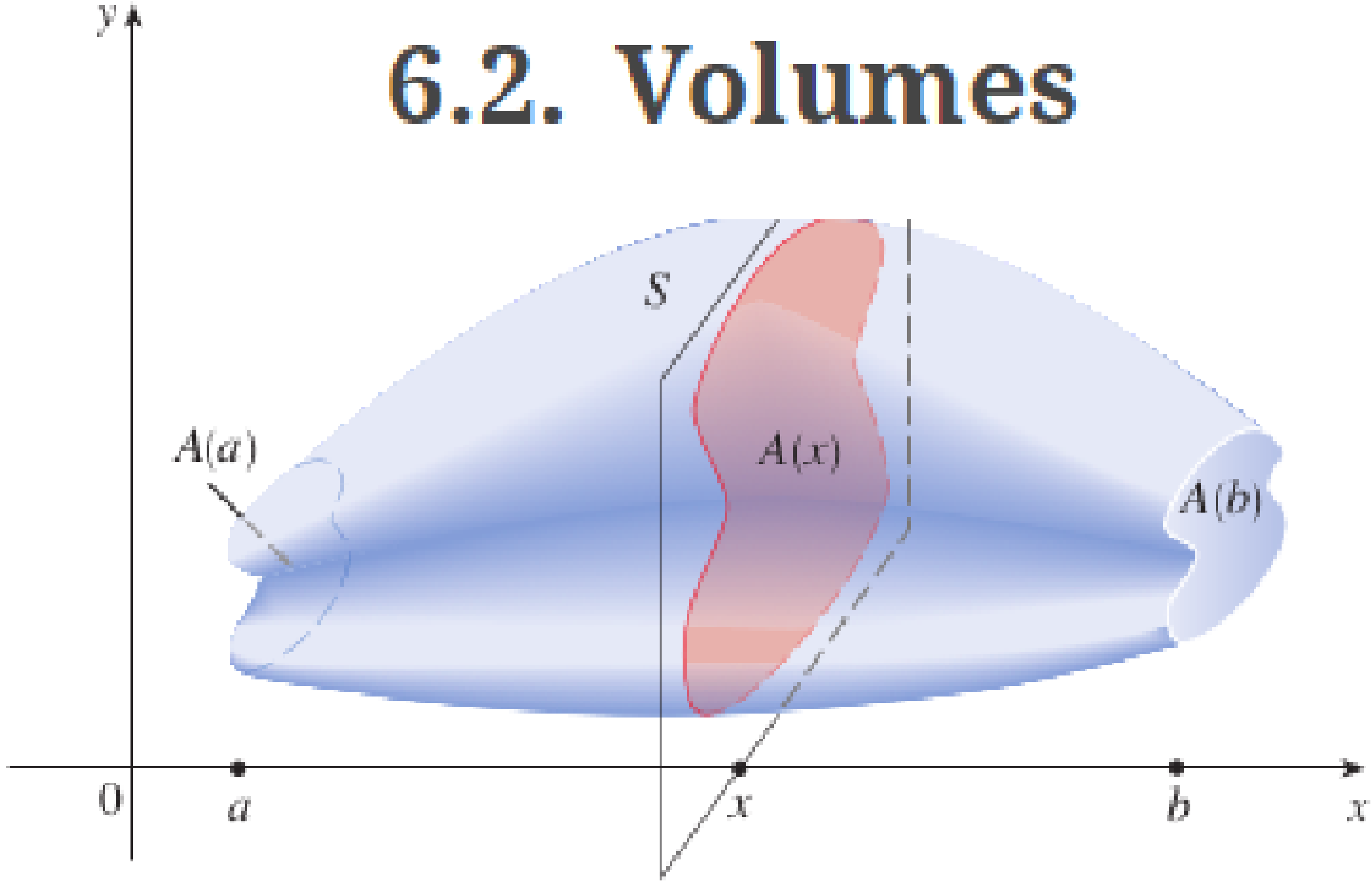
6.2. Volumes



Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x , through x and perpendicular to the x-axis , is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \, \Delta x = \int_a^b A(x) \, dx$$

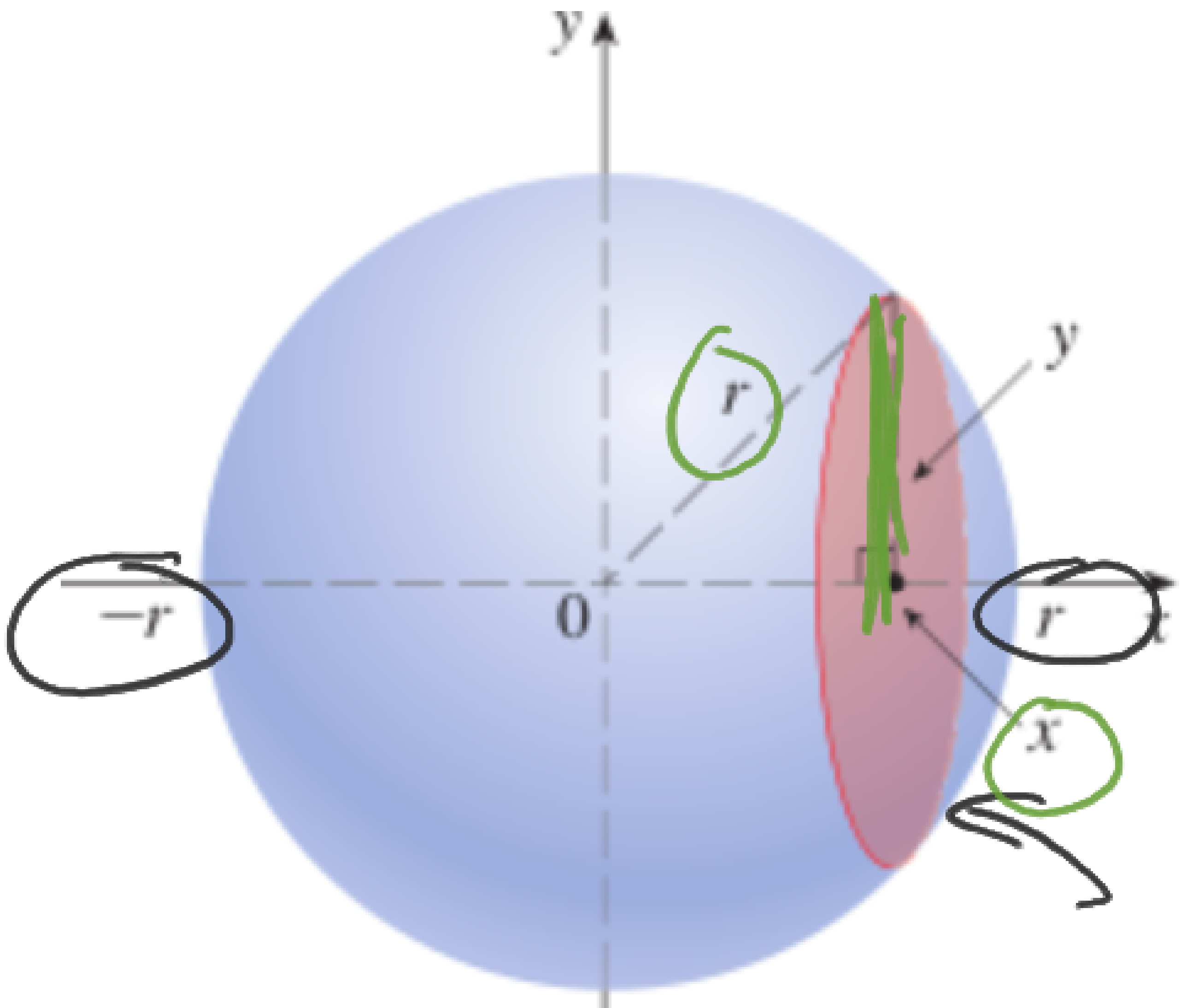
Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$

$$\int_a^b A(x)\,dx$$

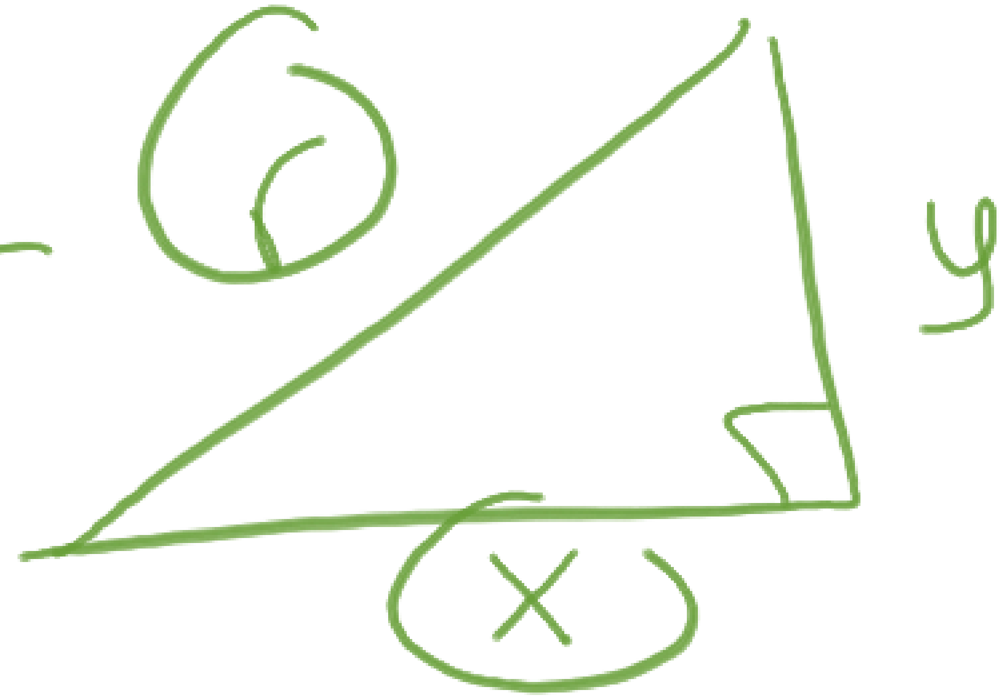
$$A(x) = \pi\left(\sqrt{r^2 - x^2}\right)^2$$

$$A(x) = \pi(r^2 - x^2)$$

$$\int_{-r}^{r} \pi(r^2 - x^2)\,dx \qquad y = \sqrt{r^2 - x^2}$$



πr^2



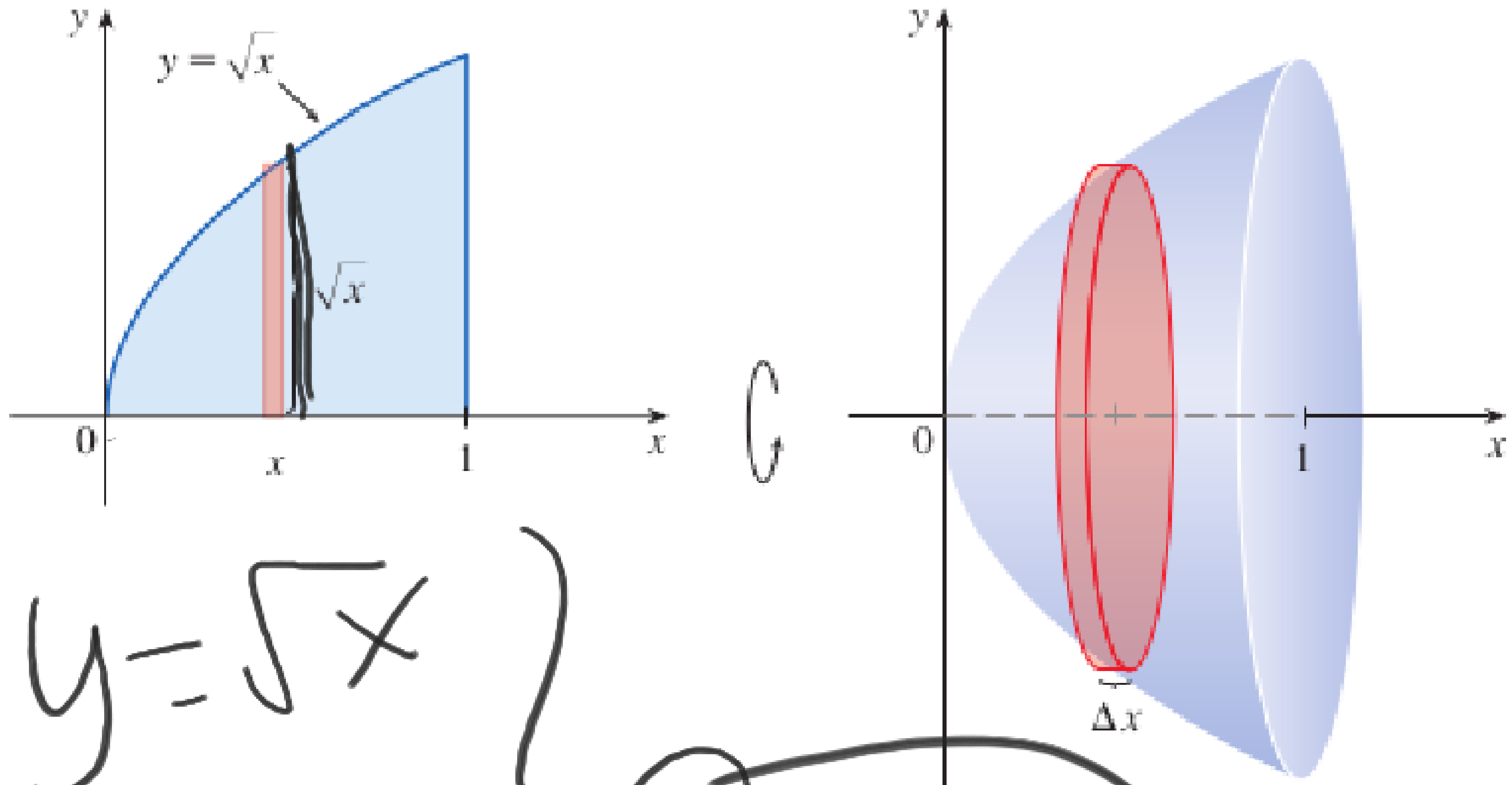
$$V = \int_{-r}^{r} \pi\left(r^2 - x^2\right) dx$$

$$V = 2\pi \int_0^r r^2 - x^2 \, dx$$

$$V = 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$V = 2\pi \left(r^3 - \frac{r^3}{3} \right) = 2\pi \left(\frac{2}{3} r^3 \right) = \boxed{\frac{4}{3}\pi r^3}$$

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.



(a) (b)

$$\int_a^b A(x)\,dx$$

$$A(x) = \pi r^2$$

$$y = \sqrt{x}$$

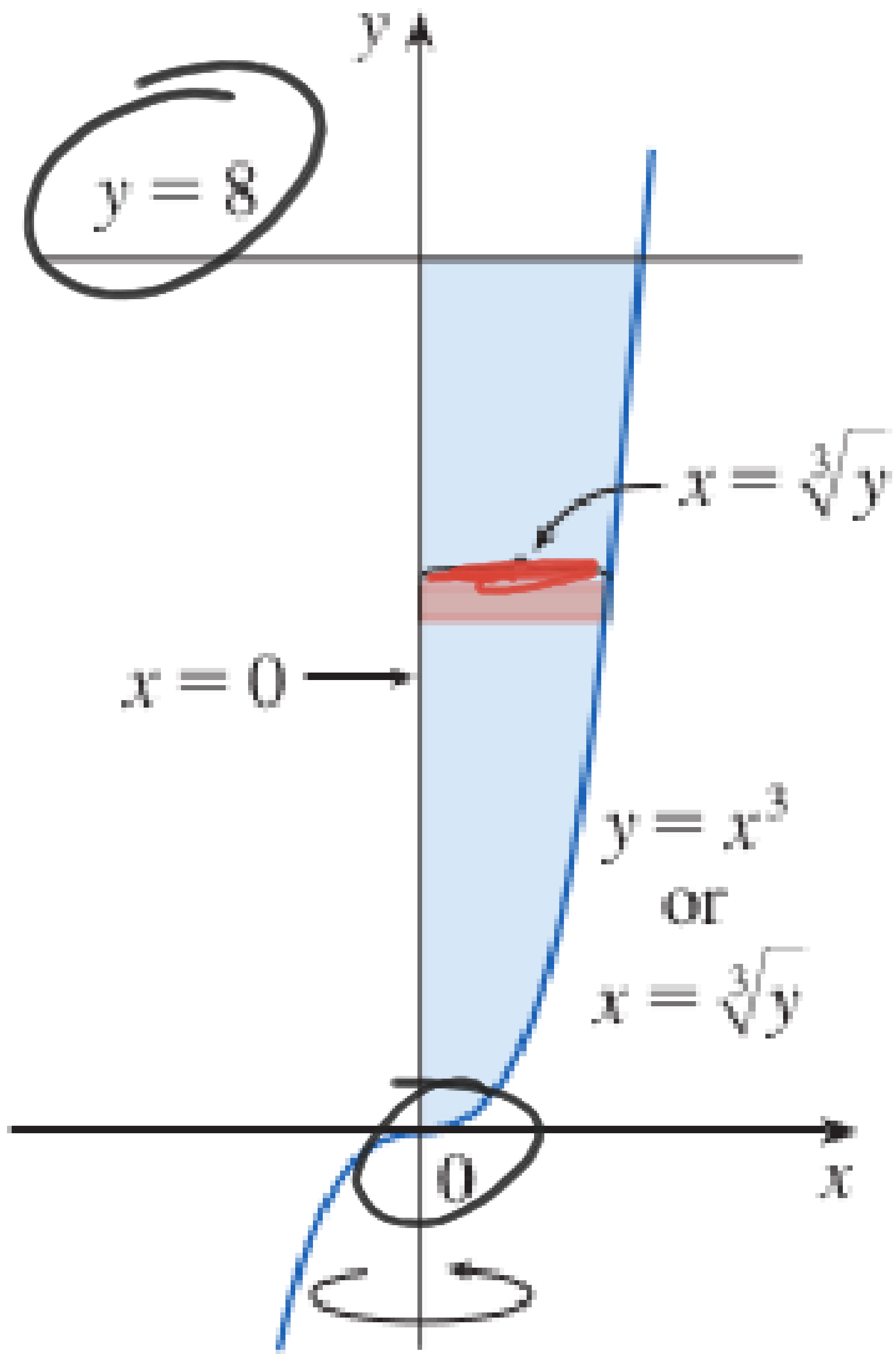
$$r = \sqrt{x}$$

$$V = \frac{\pi}{2}$$

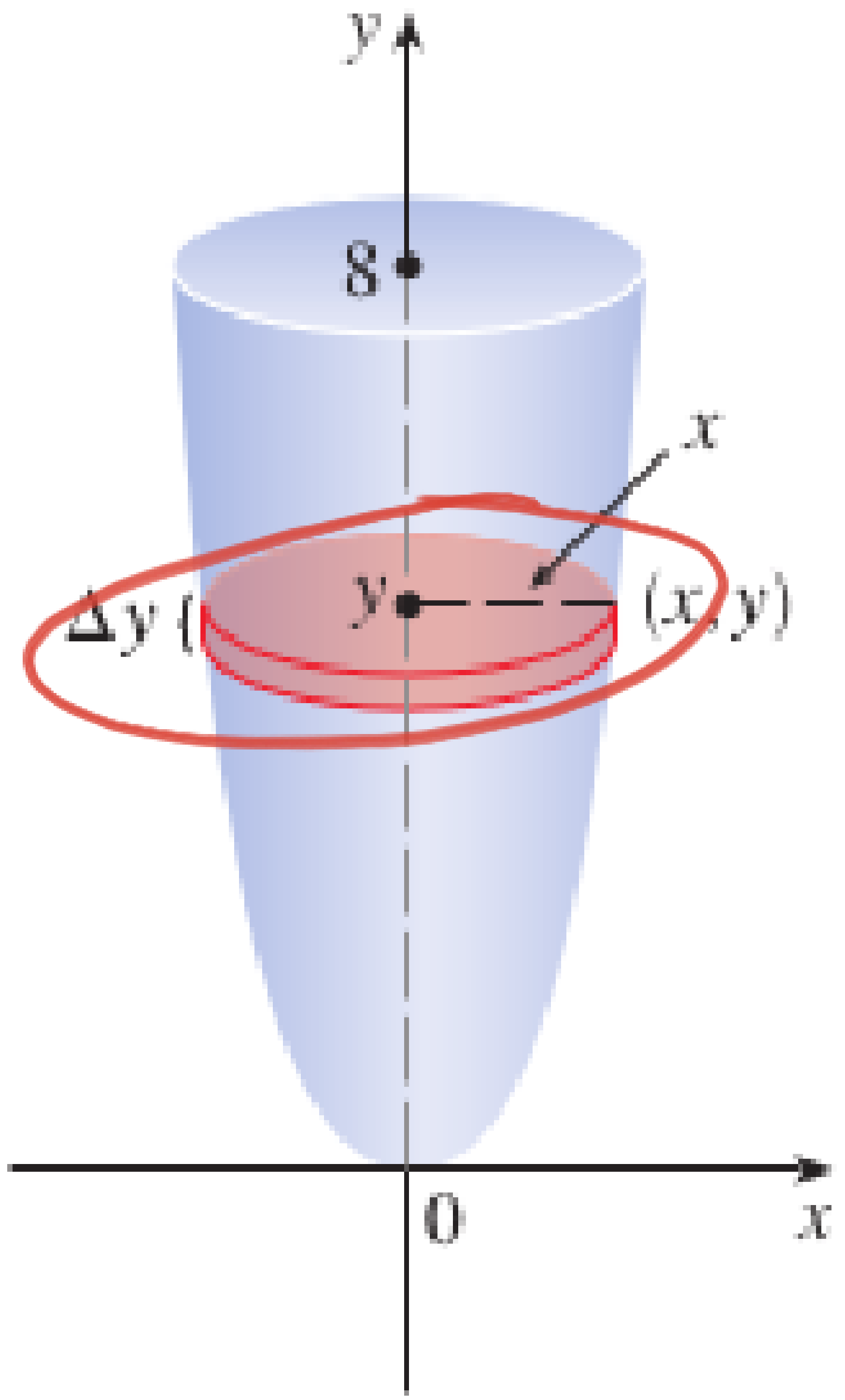
$$\int_0^1 \pi (\sqrt{x})^2$$

$$\int_0^1 \pi x\,dx = \frac{\pi x^2}{2}\Big|_0^1$$

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis .



(a)



(b)

$$y = x^3$$

$$\sqrt[3]{y} = x = r$$

$$A(y) = \pi r^2$$

$$= \pi\left(\sqrt[3]{y}\right)^2$$

$$= \pi y^{2/3}$$

$$\int_0^8 \pi y^{2/3}\, dy$$

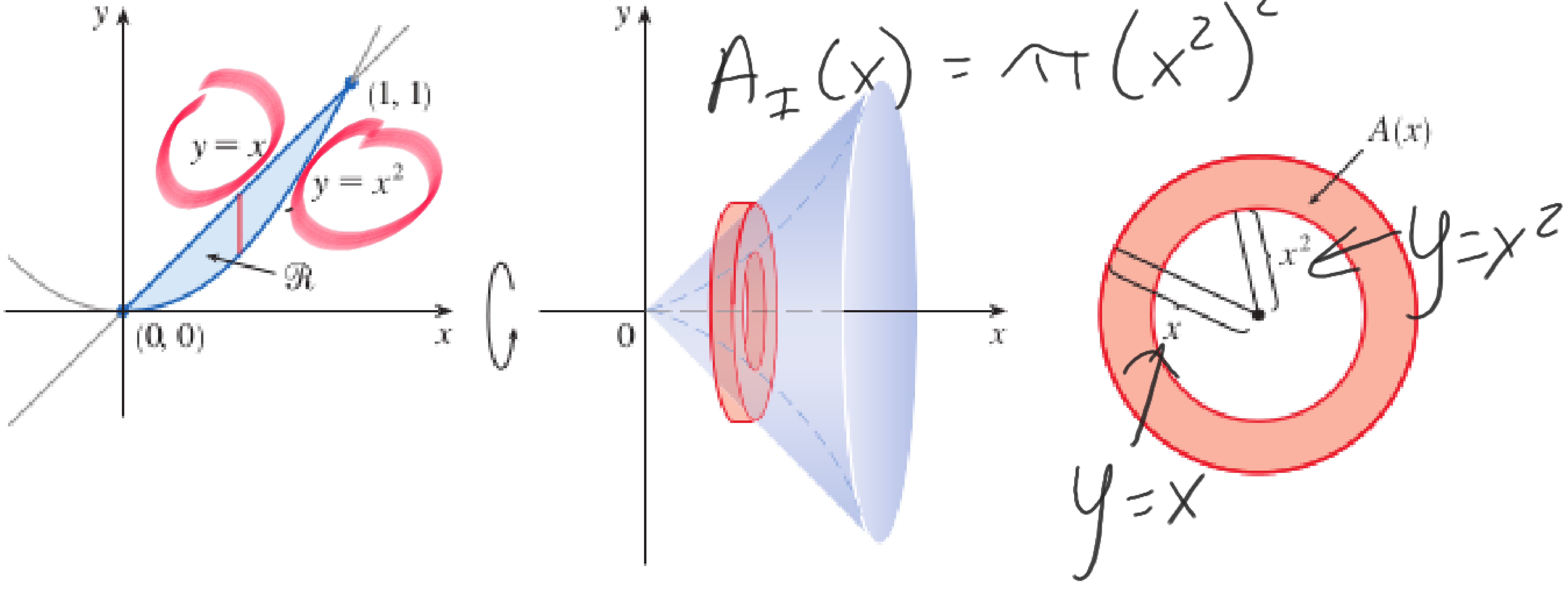
$$\int_0^8 \pi y^{2/3}\,dy = \frac{\pi y^{5/3}}{5/3} = \frac{3\pi}{5} y^{5/3}\Big|_0^8$$

$$(8)^{5/3} = 32\left(\frac{3\pi}{5}\right) = \frac{96\pi}{5}$$

The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

$$A_o(x) = \pi(x)^2$$

$$A_I(x) = \pi(x^2)^2$$



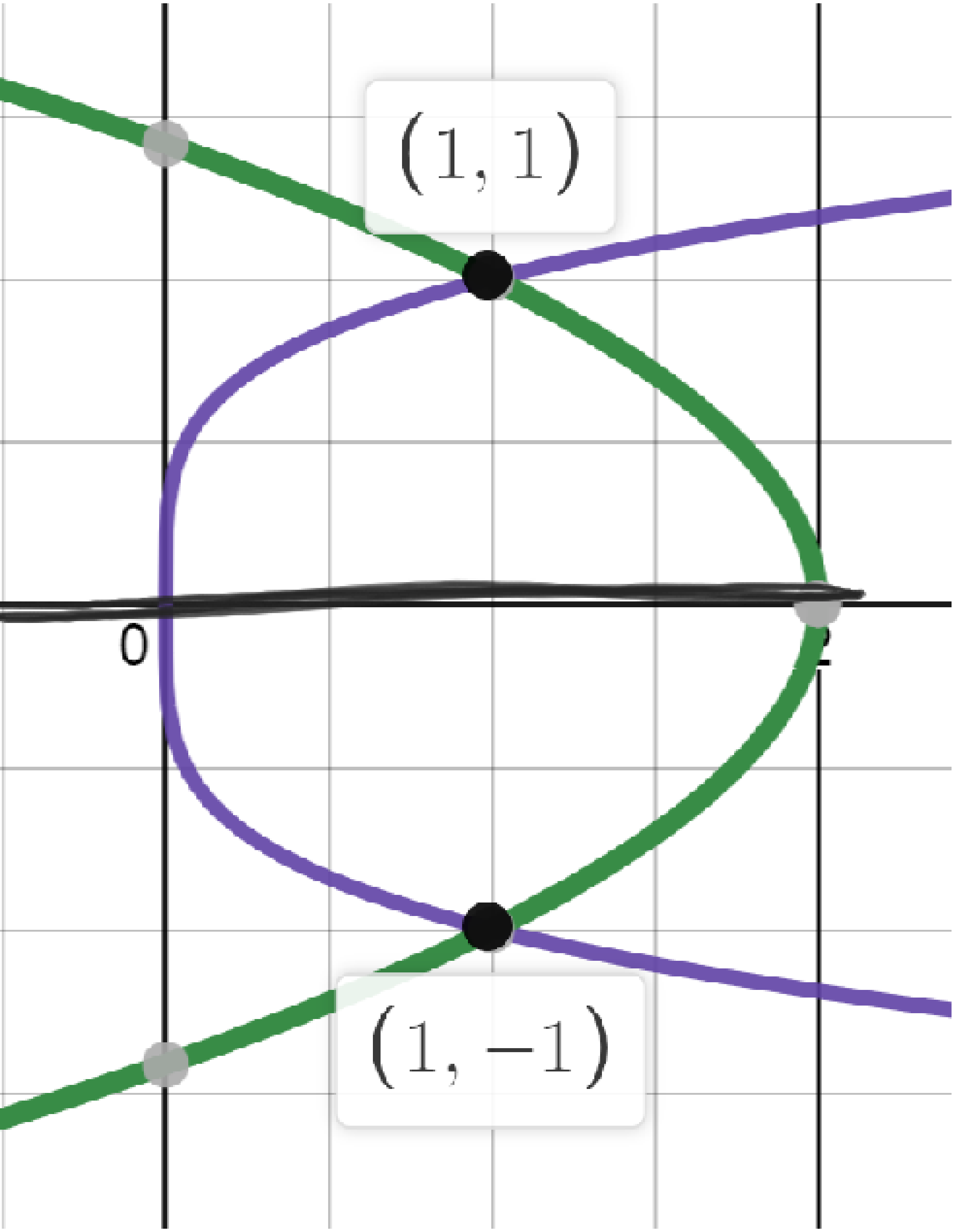
(a) (b) (c)

$$\int_0^1 \left(\pi x^2 - \pi x^4\right) dx$$

$$\pi \int_0^1 \left(x^2 - x^4\right) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \pi \left(1/3 - 1/5\right) = \boxed{\frac{2\pi}{15}}$$

20. $x = 2 - y^2$, $x = y^4$; about the y-axis



$$A_o(y) = \pi(2-y^2)^2$$

$$A_I(y) = \pi(y^4)^2$$

$$2\int_0^1 \left(\pi(2-y^2)^2 - \pi(y^4)^2\right)dy$$

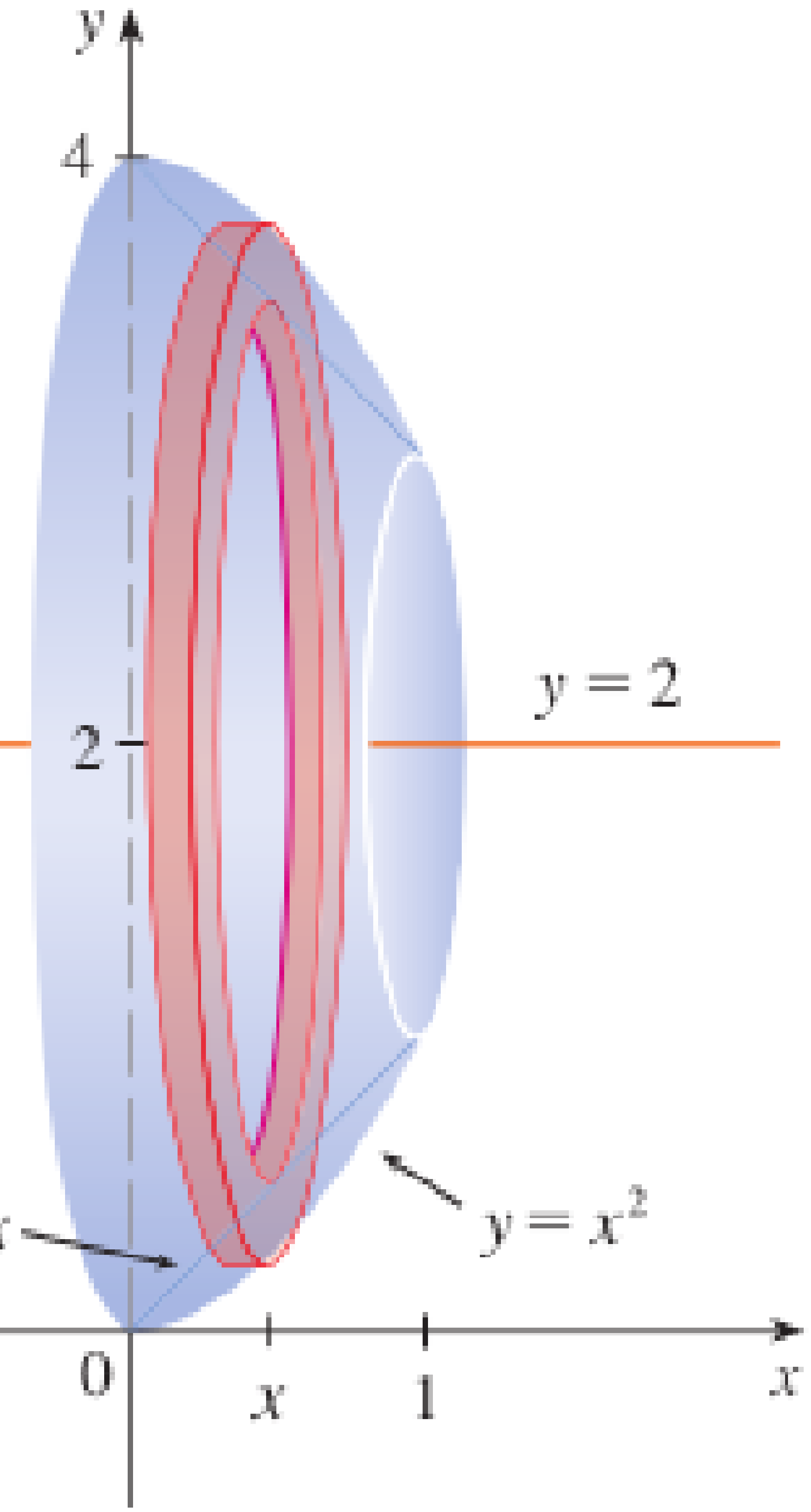
y^8

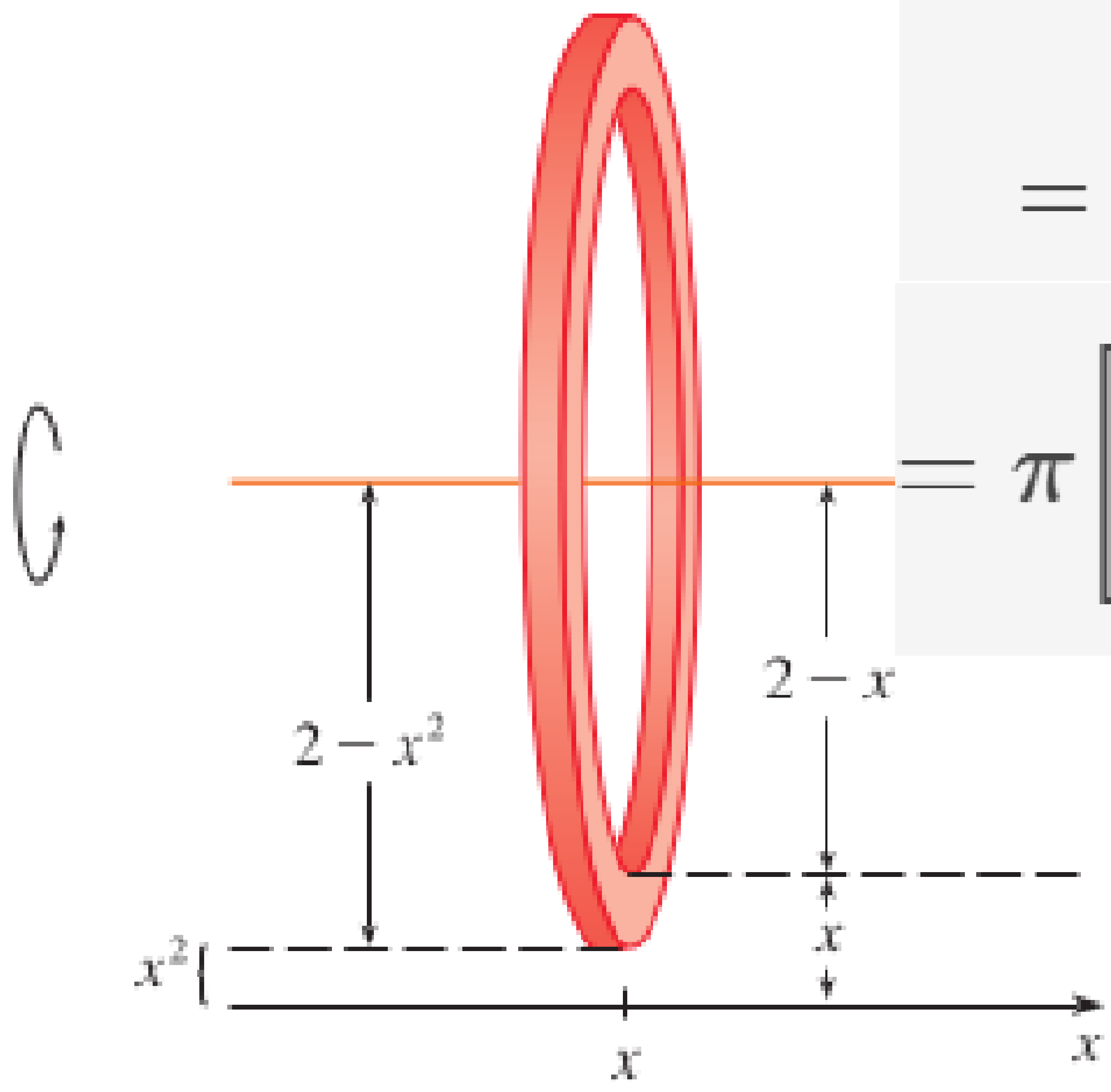
$4 - 4y^2 + y^4$

$$2\pi\int_0^1 (4-4y^2+y^4-y^8)\,dy$$

$$2\pi\int_0^1 (4-4y^2+y^4-y^8)\,dy$$

$$2\pi\left(4y-\frac{4}{3}y^3+\frac{1}{5}y^5-\frac{1}{9}y^9\right)\Big|_0^1=\frac{248}{45}\pi$$





$$= \pi \int_0^1 [(2 - x^2)^2 - (2 - x)^2]\ dx$$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4x)\ dx$$

$$= \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2} \right]_0^1 = \frac{8\pi}{15}$$

$A_0(x) = \pi(2 - x^2)^2$

$A_I(x) = \pi(2 - x)^2$

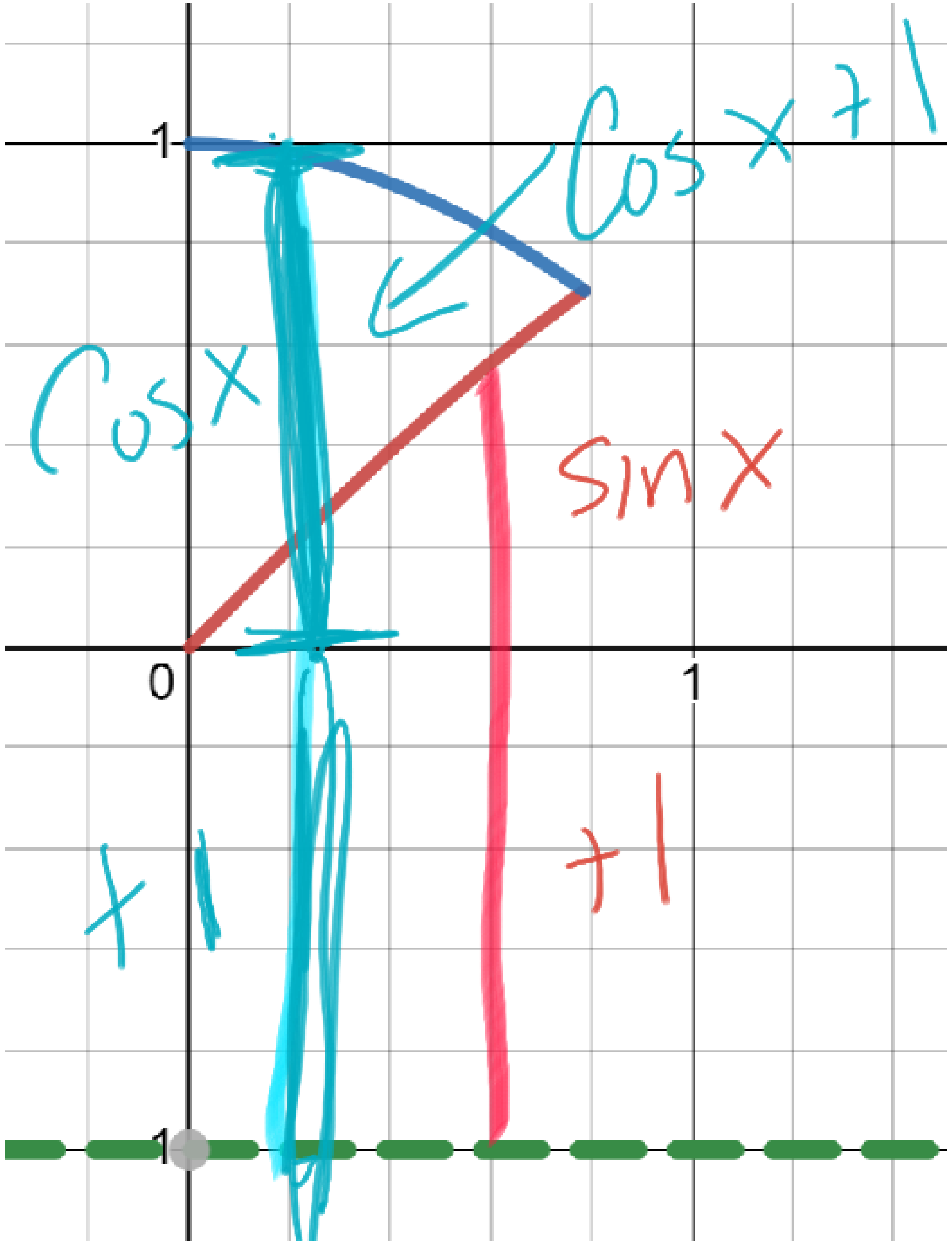
24. $y = \sin x\ ,\quad y = \cos x\ ,\quad 0 \leqslant x \leqslant \pi/4;\qquad$ about $\ y = -1$

$$A_o(x) = \pi(\cos x + 1)^2$$

$$A_I(x) = \pi(\sin x + 1)^2$$

$\cos^2 x + 2\cos x + 1$

$\sin^2 x + 2\sin x + 1$



$$\pi \int_0^{\pi/4} (\cos^2 x + 2\cos x + 1) - (\sin^2 x + 2\sin x + 1)$$

$$\pi \int_0^{\pi/4} \left(\underbrace{\boxed{\cos^2 x - \sin^2 x}}_{= \cos 2x} + 2\cos x - 2\sin x \right) dx$$

$$\pi \left[\frac{1}{2} \sin 2x + 2\sin x + 2\cos x \right]_0^{\pi/4}$$

$u = 2x$

$du = 2\,dx \qquad \frac{1}{2} du = dx$

$\cos 2x$ $\qquad \frac{1}{5x}$

$\frac{1}{2} \sin 2x$ $\qquad \frac{1}{5} \ln(5x)$

$$\pi\left[\frac{1}{2}\sin 2x + 2\sin x + 2\cos x\right]_0^{\pi/4}$$

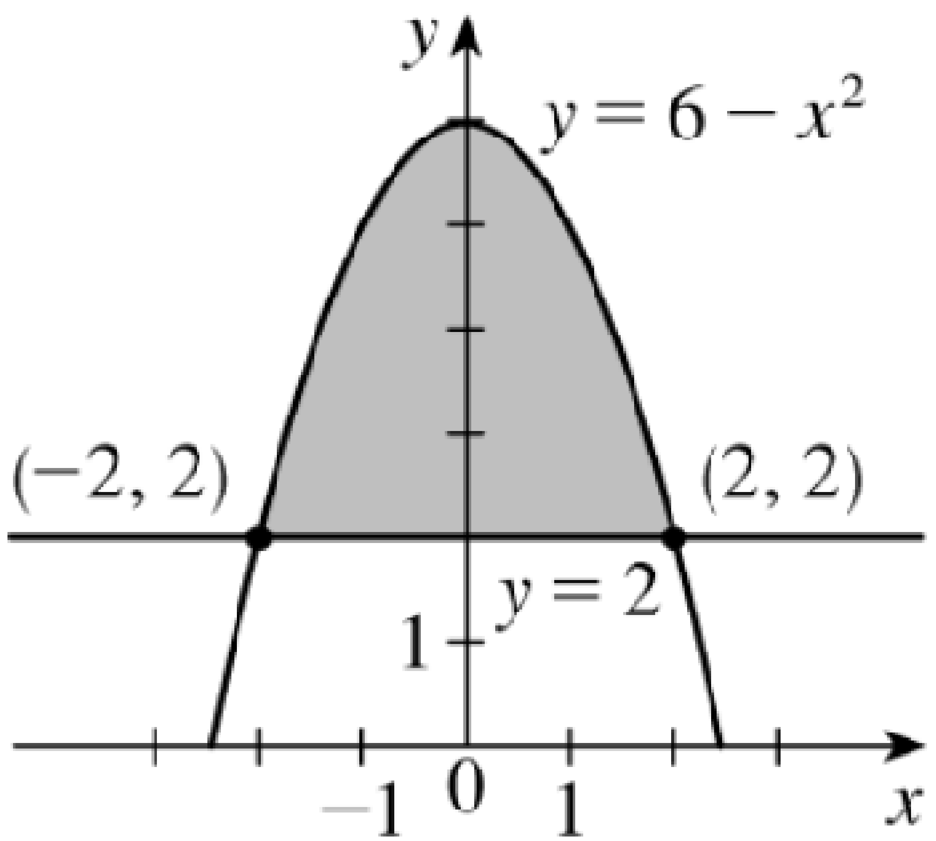
$$\pi\left[\frac{1}{2}(1) + 2\left(\frac{\sqrt{2}}{2}\right) + 2\left(\frac{\sqrt{2}}{2}\right)\right] - \left[0 + 0 + 2\right]$$

$$\pi\left(\frac{1}{2} + \sqrt{2} + \sqrt{2} - 2\right) = \boxed{\left(-\frac{3}{2} + 2\sqrt{2}\right)\pi}$$

18. $y = 6 - x^2$, $y = 2$; about the x-axis

$$A_o(x) = \pi(6 - x^2)^2$$

$$A_I(x) = \pi(2)^2$$



2 and outer radius $6 - x^2$, so its area is

$$A(x) = \pi[(6 - x^2)^2 - 2^2] = \pi(x^4 - 12x^2 + 32).$$

$$2\int_0^2 \pi(x^4 - 12x^2 + 32)\,dx$$

$$= 2\pi\left[\tfrac{1}{5}x^5 - 4x^3 + 32x\right]_0^2$$

$$= 2\pi\left(\tfrac{32}{5} - 32 + 64\right) = 2\pi\left(\tfrac{192}{5}\right) = \tfrac{384\pi}{5}$$

$$(6-x^2)^2 = (6-x^2)(6-x^2)$$

$$= 36 - 6x^2 - 6x^2 + x^4 - 4$$

$$x^4 - 12x^2 + 32$$