series for convergence or divergence.

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{3n}}{(2n)!}$$

Ratio Test

$$\lim_{n\to\infty} \frac{\pi^{3(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{\pi^{3n}} = \frac{\pi^3}{(2n+2)(2n+1)} \to 0$$

$$\frac{\pi^{3n+3}}{\pi^{3n}} = \frac{\pi^{3n}\cdot\pi^3}{\pi^{3n}} = \pi^3$$

Convergent by Ratio Test

$$\frac{2n!}{(2n+2)!} = \frac{2n!}{(2n+2)(2n+1)(2n!)} = \frac{1}{(2n+2)(2n+1)}$$

Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{\sin(6n)}{1+8^n}$$

$$\frac{\sin(6n)}{1+8^n} < \frac{1}{1+8^n} < \frac{1}{8^n} = \left(\frac{1}{8}\right)^n$$

$\left(\frac{1}{8}\right)^n$ is convergent

$r = \frac{1}{8} < 1$ Geometric series

So $\sum_{n=1}^{\infty} \frac{\sin 6n}{1+8^n}$ is convergent by direct comparison

Test the series for convergence or divergence.

$$\sum_{k=1}^{\infty} \frac{k \ln(k)}{(k+5)^3}$$

$$\frac{k \ln(k)}{(k+5)^3} < \frac{k \ln(k)}{k^3} = \frac{\ln(k)}{k^2}$$

$$\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln(x)}{x} - \frac{1}{x}\right]_1^t \text{ [using integration by parts]} = 1$$

So $\frac{\ln(k)}{k^2}$ is convergent then

$\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+5)^3}$ is convergent by direct convergence

Find the radius of convergence, $R$, of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{\sqrt{n}} x^n$$

If $a_n = \frac{(-1)^n 3^n}{\sqrt{n}} x^n$, then $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} 3^{n+1} x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n \; {}^n x^n} \right| = \lim_{n \to \infty} \sqrt{\frac{n}{n+1}} \cdot \; |x| = 3\,|x|.$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{\sqrt{n}} x^n$ converges when $3|x| < 1 \iff |x| < \frac{1}{3}$, so $R = \frac{1}{3}$.

When $x = \frac{1}{3}$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test.

When $x = -\frac{1}{3}$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a $p$-series $\left(p = \frac{1}{2} \le 1\right)$.

Thus, the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$.

Evaluate the indefinite integral as an infinite series.

$$\int \frac{\cos(x) - 1}{x}\,dx$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow$$

$$\cos(x) - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow$$

$$\frac{\cos(x) - 1}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!} \Rightarrow$$

$$\int \frac{\cos(x) - 1}{x}\,dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n \cdot (2n)!}, \text{ with } R = \infty$$