1. **Test for Divergence** If you can see that $\lim_{n\to\infty} a_n$ may be different from 0, then apply the Test for Divergence.
2. ***p*-Series** If the series is of the form $\Sigma\, 1/n^p$, then it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
3. **Geometric Series** If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, then it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
4. **Comparison Tests** If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply a comparison test to $\Sigma\, |a_n|$ and test for absolute convergence.

5. **Alternating Series Test** If the series is of the form $\Sigma(-1)^{n-1}b_n$ or $\Sigma(-1)^n b_n$, then the Alternating Series Test is an obvious possibility. Note that if Σb_n converges, then the given series is absolutely convergent and therefore convergent.

6. **Ratio Test** Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. **Root Test** If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. **Integral Test** If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

$$\sum_{n=1}^{\infty} \boxed{\frac{n-1}{2n+1}}$$

$$\lim_{n\to\infty} \frac{n-1}{2n+1} \frac{/n}{/n}$$

$$\lim_{n\to\infty} \frac{1-1/n}{2+1/n} = \frac{1}{2} \neq 0$$

Diverges by Test for Divergence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}\,/n^3}{3n^3+4n^2+2\,/n^3} \qquad \frac{\sqrt{1/n^3+1/n^6}}{3+4/n+2/n^3} \begin{matrix} \to 0 \\ \to 3 \end{matrix} \quad \frac{0}{3} \to 0$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \to \frac{\sqrt{n^3+1}}{3n^3} \Rightarrow \frac{\sqrt{n^3}}{3n^3} \to \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

$$\boxed{\begin{matrix} P = \frac{3}{2} > 1 \\ \text{Convergent} \end{matrix}}$$

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = 1 \qquad a_n \to \text{convergent}$$

$$\lim_{n\to\infty} \left| \frac{\dfrac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{1/3n^{3/2}} \right| = \frac{(\sqrt{n^3+1})(3n^{3/2})}{3n^3+4n^2+2} = \frac{3n^3\sqrt{1+1/n^3}}{3n^3+4n^2+2}\Big/ n^3 = \frac{3\sqrt{1+1/n^3}}{3+4/n+2/n^3} \to \frac{3}{3} = 1$$

$$= \sqrt{n^3(1+1/n^3)}\,(3n^{3/2})$$

$$n^{3/2}\sqrt{1+1/n^3} \cdot 3n^{3/2} = 3n^3\sqrt{1+1/n^3}$$

$$\sum_{n=1}^{\infty} \boxed{(-1)^n} \boxed{\frac{n^2}{n^4+1}}$$

1) $b_{n+1} \le b_n$

2) $\lim_{n\to\infty} b_n = 0$

Convergent AST

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies the conditions

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

1) $\frac{(n+1)^2}{(n+1)^4+1} < \frac{n^2}{n^4+1}$ ✓

2) $\lim_{n\to\infty} \frac{n^2}{n^4+1} \Big/ n^4 \to 0$

$$\sum_{n=0}^{\infty} \frac{n^2}{n^4+1}$$

$\frac{n^2}{n^4+1} < \frac{n^2}{n^4} = \frac{1}{n^2}$ $p=2>1$

converges

converges

$$\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{n^4+1}$$ Absolutly Convergent

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$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \begin{array}{ll} <1 & \text{Conv} \\ >1 & \text{DIV} \\ =1 & \text{IDK} \end{array}$$

$$\lim_{n\to\infty} \left| \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} \right| = \left| \frac{\cancel{k!}}{(k+1)(\cancel{k!})} \cdot \frac{(\cancel{2^k})(2)}{\cancel{2^k}} \right|$$

$$\lim_{n\to\infty} \left| \frac{2}{k+1} \right| \to 0 < 1$$

Convergent by Ratio Test

$$\sum_{n=1}^{\infty} ne^{-n^2}$$ converges

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$$f(x) = xe^{-x^2} = \frac{x}{e^{x^2}}$$

$$\int_1^{\infty} xe^{-x^2}\,dx$$

$$u = -x^2$$

$$du = -2x\,dx$$

$$-\frac{1}{2}du = x\,dx$$

$$-\frac{1}{2}\int_1^{\infty} e^u\,du = -\frac{1}{2}e^u\Big]_1^{\infty}$$

$$= -\frac{1}{2}\Big[e^{-x^2}\Big]_1^{\infty}$$

converges

$$= -\frac{1}{2}\Big[0 - \frac{1}{e}\Big] = \boxed{\frac{1}{2e}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n} < \frac{1}{3^n} = \boxed{\left(\frac{1}{3}\right)^n \quad r = \frac{1}{3} < 1 \quad \text{Convergent}}$$

Convergent

Direct Comparison

to $\left(1/3\right)^n$

$$\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$$

1) $b_{n+1} < b_n$

2) $\lim_{n \to \infty} b_n = 0$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) \to \cos(0) = 1$$

Diverges

Test for Divergence

$$\sum_{n=1}^{\infty} \frac{4 - \cos n}{\sqrt{n}} > \frac{4-1}{\sqrt{n}} = \frac{3}{n^{1/2}} \qquad P = \frac{1}{2} < 1$$

Diverges

Diverges

Direct Comparison