9) Find the radius and interval of convergence for the following problem:

$$\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}}{(2n-1)2^n}(x-1)^n$$

Ratio Test →

$$\left|\frac{a_{n+1}}{a_n}\right|$$

$$\left|\frac{(x-1)^{n+1}}{(2(n+1)-1)(2^{n+1})} \cdot \frac{(2n-1)(2^n)}{(x-1)^n}\right|$$

$2n+2-1$

$$\left|\frac{\cancel{(x-1)^n}(x-1)}{\cancel{(x-1)^n}} \cdot \frac{(2n-1)}{(2n+1)} \cdot \frac{\cancel{2^n}}{(\cancel{2^n})(2)}\right|$$

9) Find the radius and interval of convergence for the following problem:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$$

$$\boxed{-1 < x \le 3} \quad \boxed{(-1, 3]}$$

$\cancel{(-1)^n}$ $\cancel{2^n}$

$x = -1$

$$\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}}{(2n-1)\cancel{2^n}}(-2)^n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$\lim_{n\to\infty} \left| \frac{1/(2n-1)}{1/n} \right|$$

$$\frac{1}{n} \overset{?}{<} \frac{1}{2n-1}$$

D D

$$\lim_{n\to\infty} \frac{n}{2n-1} \to \frac{1}{2}$$

$x = 3$

$$\sum^{\infty} \frac{(-1)^n}{(2n-1)(2^n)}(2^n) = \sum \frac{(-1)^n}{(2n-1)}$$

$$\frac{1}{2(n+1)-1} < \frac{1}{2n-1}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$x^2\sin(x^2) = x^2\sum_{n=0}(-1)\frac{x}{(2n+1)!} = \sum_{n=0}(-1)\frac{}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+4}}{(2n+1)!}$$

$$\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+4}}{(2n+1)!}\,dx =$$

$$\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+5}}{(2n+1)!\,(4n+5)}}$$

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$$

$$\frac{n-1}{n^3+1} \overset{?}{<} \frac{1}{n^2}$$

P-Series

C $\quad p = 2 > 1$

$$\boxed{\frac{n-1}{n^3+1} < \frac{n-1}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}}$$

Direct Comparison

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^3+1}}{1/n^2} = \frac{n^3-n^2}{n^3+1} \Big/ n^3 = \frac{1-0}{1+0} = 1 > 0$$

Convergent

L.C.T.

2) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}} \rightarrow \sum_{n=1}^{\infty} \left(\frac{n^2}{(1+n)^3}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2}{(1+n)^3}\right)^n} \rightarrow \lim_{n\to\infty} \frac{n^2}{(1+n)^3} \Big/ n^3$$

$$\frac{n^2/n^3}{\left(\frac{1+n}{n}\right)^3} = \frac{0}{(1)^3} = 0 < 1$$

Convergent by Root Test

3) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

$\ln x = u$

$\frac{1}{x} = du$

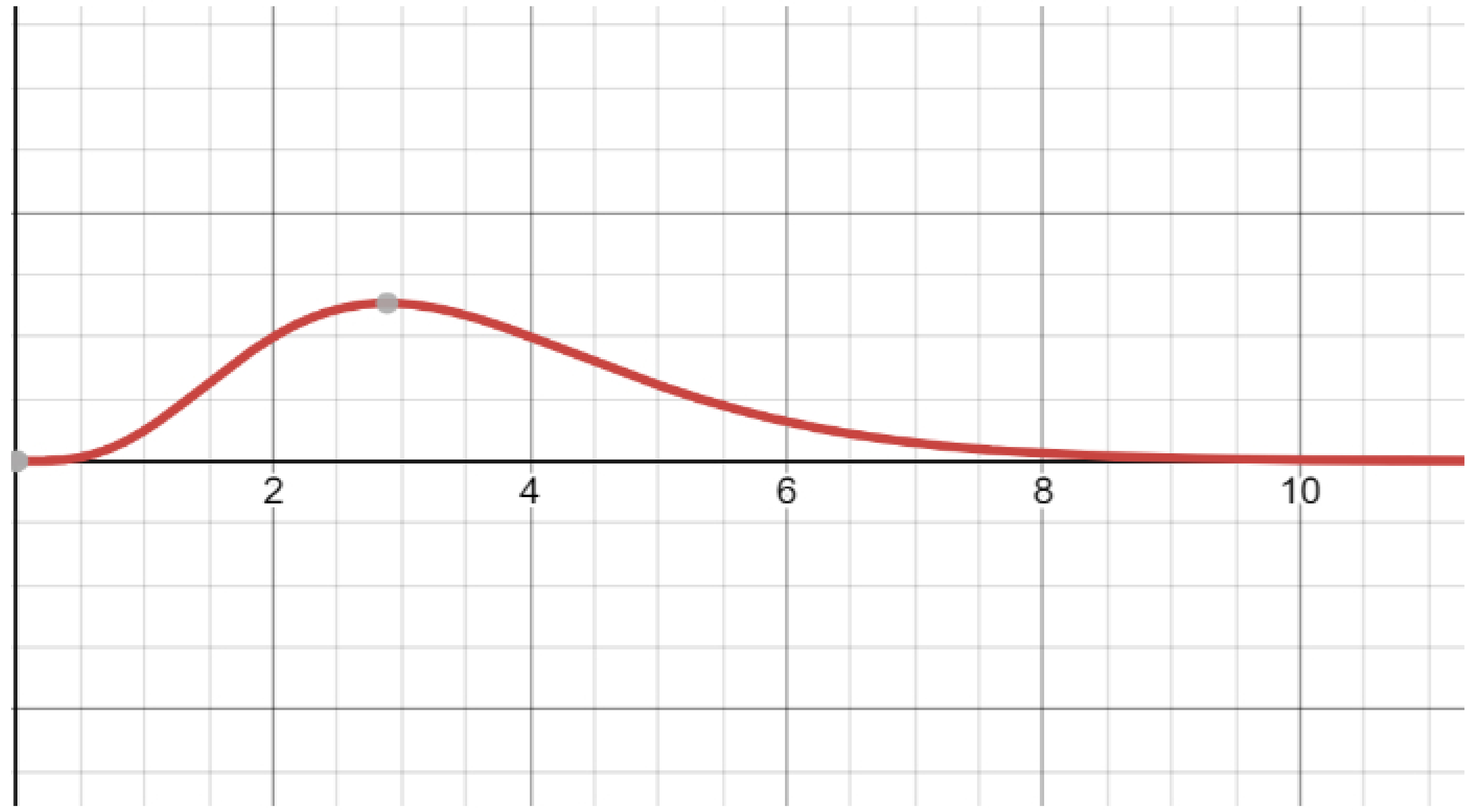
$$\int_2^{\infty} \frac{1}{x(\ln x)^{1/2}}$$

$$\int_2^{\infty} u^{-1/2}\, du \to 2u^{1/2} = 2\sqrt{\ln x}\Big]_2^{\infty} \to \infty$$

Divergent by Integral Test

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$$

1) $\frac{(n+1)^4}{4^{n+1}} < \frac{n^4}{4^n}$ ✓



2) $\lim\limits_{n\to\infty}\frac{n^4}{4^n} = 0$ ✓

Convergent
A.S.T.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{(n+1)^4}{4^{n+1}}\cdot\frac{4^n}{n^4}\right|$$

4^{n+1}

$4^n\cdot 4^1$

$$\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^4\cdot\frac{4^n}{4^n\cdot 4}\right|$$

Divergent

5) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n} \rightarrow \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \boxed{\frac{1}{n}} \rightarrow$ Diverges

$$\lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^3+n} \Big/ n^3 = \frac{\sqrt{n^4/n^6 + 1/n^6}}{1 + 1/n^2} = \frac{0}{1} = 0$$

LCT

$$\lim_{n\to\infty} \left| \frac{\frac{\sqrt{n^4+1}}{n^3+n}}{1/n} \right| = \frac{n\sqrt{n^4+1}}{n^3+n} = \frac{\sqrt{n^4+1}}{n^2+1} \Big/ n^2$$

$$= \frac{\sqrt{1+1/n^4}}{1+1/n^2} \rightarrow 1$$

6) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

1) $\frac{1}{\sqrt{n+1}-1} < \frac{1}{\sqrt{n}-1}$ ✓

$b_n = \frac{1}{\sqrt{n}-1}$

2) $\lim_{n \to \infty} \frac{1}{\sqrt{n}-1} = 0$ ✓

$$\frac{1/\sqrt{n}}{1 - 1/\sqrt{n}} = \frac{0}{1} = 0$$

Convergent by A.S.T.

7) $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

$$\frac{\sin 2n}{1+2^n} \le \frac{1}{1+2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

Convergent by Direct Comparison

$r = 1/2 < 1$

Convergent

8) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$ Diverges by L.C.T.

$$\boxed{\frac{n^2-1}{n^3+1} < \frac{n^2-1}{n^3} < \frac{n^2}{n^3} = \frac{1}{n}}$$

diverges

LCT

$$\lim_{n\to\infty} \frac{n^2-1}{n^3+1} \cdot \frac{n}{1} = \frac{n^3-n}{n^3+1} \Big/ n^3 = \frac{1-0}{1+0} = 1$$

EXAMPLE 2 Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 11.3.4, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geqslant 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \geqslant 3$$