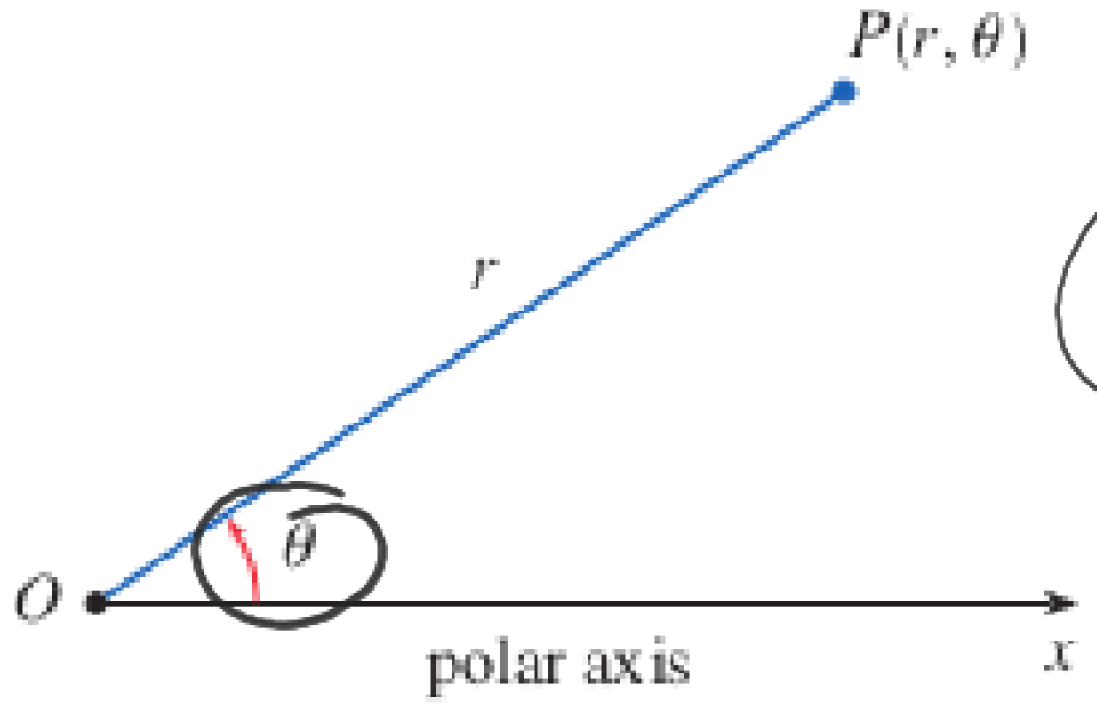
Figure 1



(r, θ)

radius angle

Details

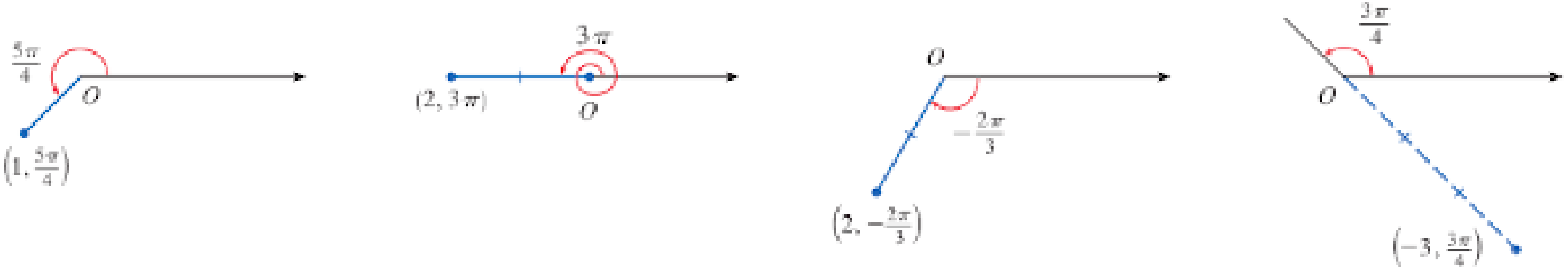
Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$

(b) $(2, 3\pi)$

(c) $(2, -2\pi/3)$

(d) $(-3, 3\pi/4)$

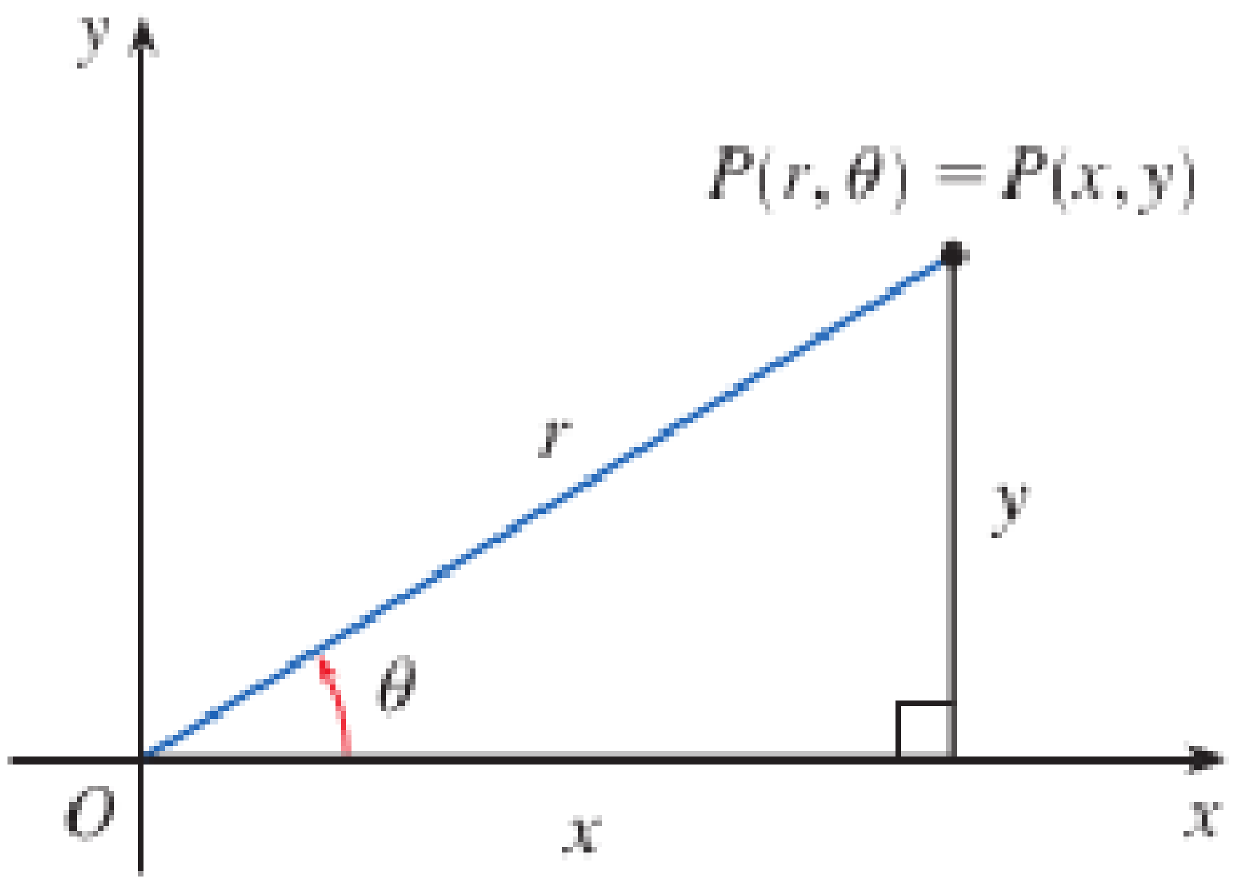


(a) (b) (c) (d)

Polar → (x,y)

$$x = r\cos\theta \qquad y = r\sin\theta$$

igure 5



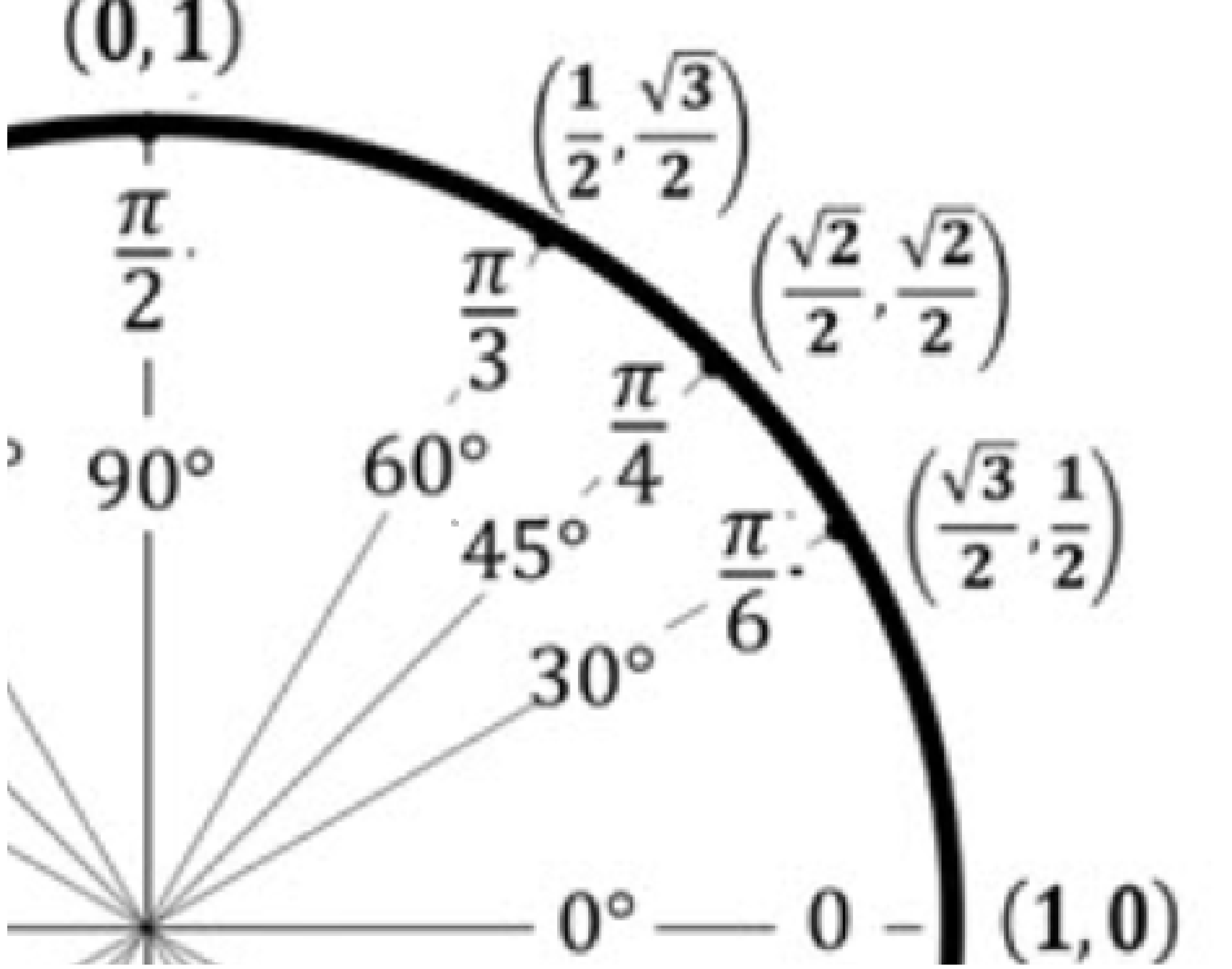
▸ Details

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

→ (x,y) to Polar

Example 2

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.



$(2, \pi/3) \to (1, \sqrt{3})$

$x = r\cos\theta$

$y = r\sin\theta$

$x = 2(\cos \pi/3)$

$y = 2(\sin \pi/3)$

$x = 2(1/2)$

$y = 2(\sqrt{3}/2)$

$x = 1$

$y = \sqrt{3}$

Example 3

Represent the point with Cartesian coordinates (1, −1) in terms of polar coordinates.

$(x, y) \rightarrow (\sqrt{2}, 7\pi/4)$

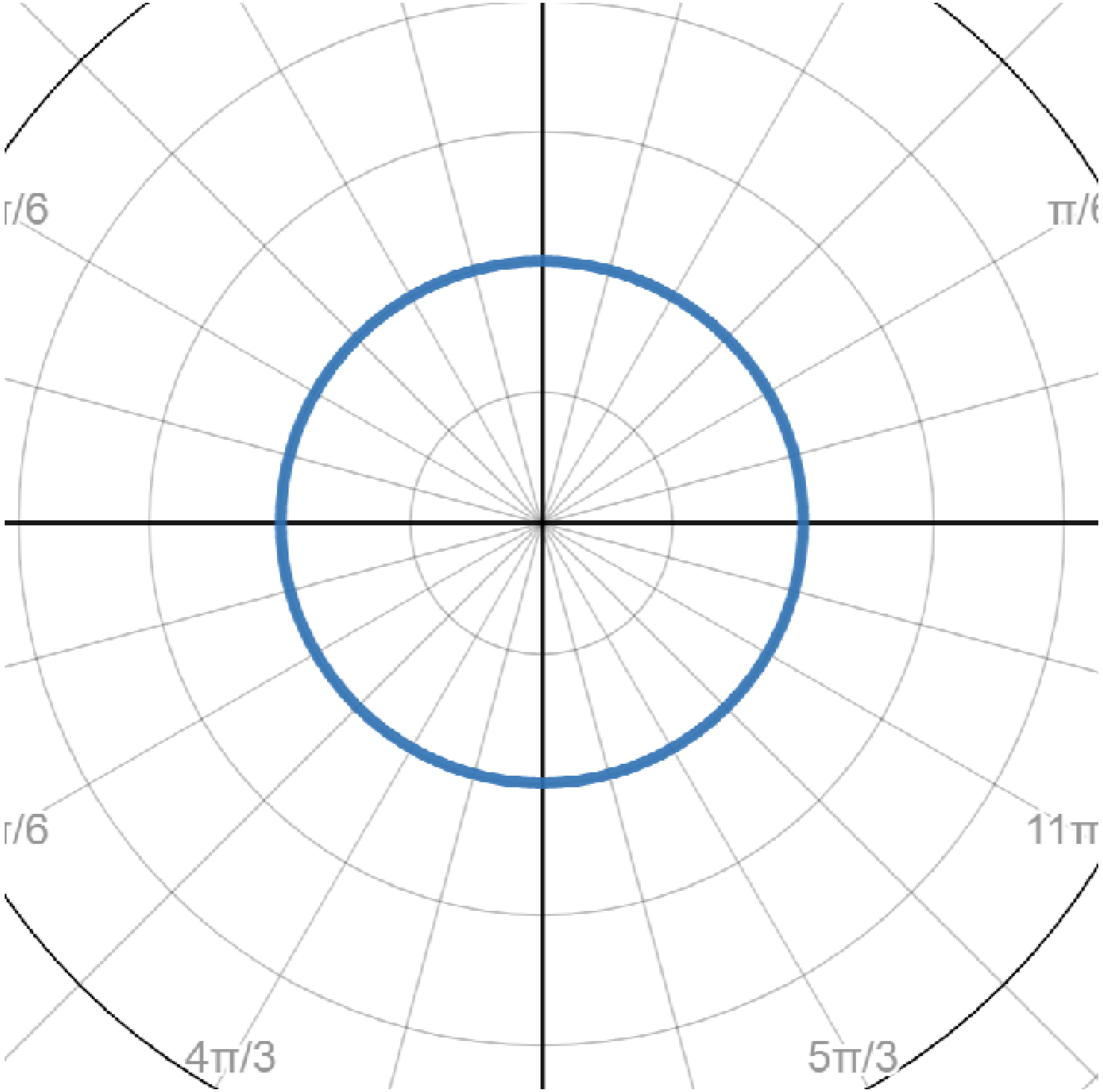
$r =$

$r^2 = x^2 + y^2 \rightarrow r^2 = (1)^2 + (-1)^2$ $\sqrt{2}$

$\tan\theta = \frac{y}{x} \rightarrow \tan\theta = \frac{-1}{1}$

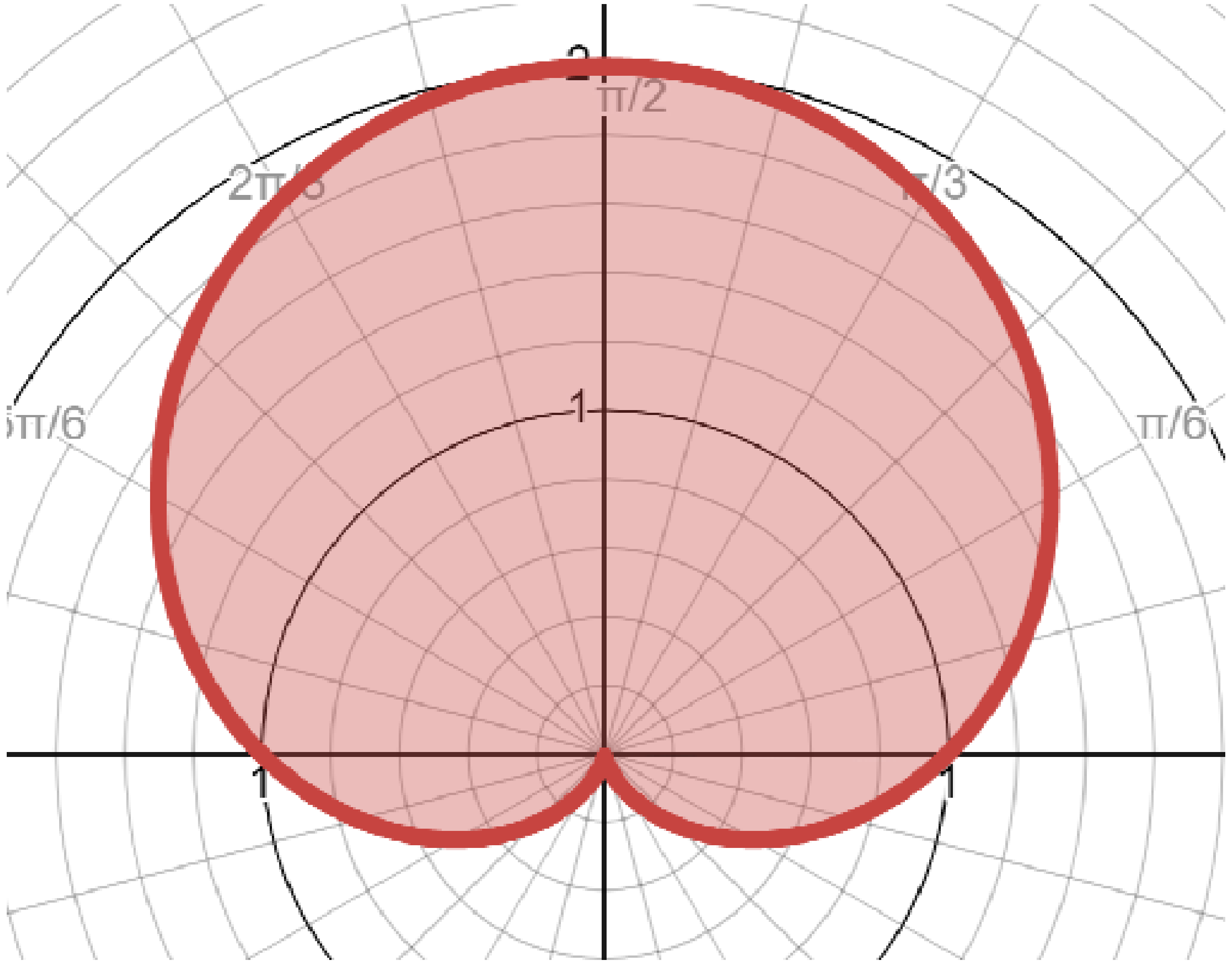
$\theta = \tan^{-1}(-1) = 7\pi/4$

What curve is represented by the polar equation $r = 2$?



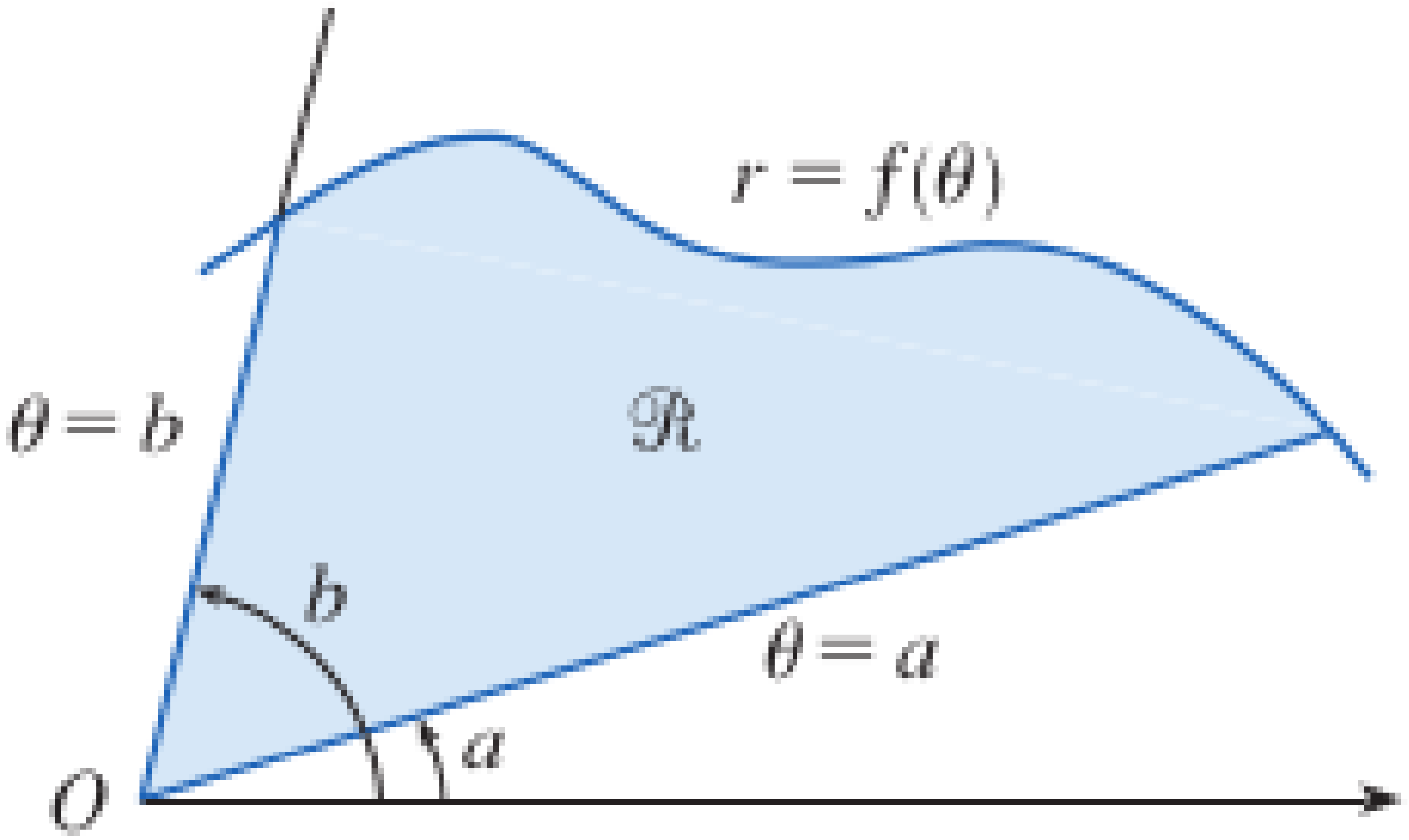
Example 7

Sketch the curve $r = 1 + \sin\theta$.



Example 8

Sketch the curve $r = \cos 2\theta$.



$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Example 1

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$\boxed{\frac{\pi}{8}}$

$r = \cos 2\theta \qquad 0 \to \frac{\pi}{4}$

$2\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta \; d\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/4} 1 + \cos 4\theta \, d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$

Example 2

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

$$3\sin\theta = 1 + \sin\theta$$

$$2\sin\theta = 1 \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin\theta = 1/2$$

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2 d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2 d\theta$$

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2 d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2 d\theta$$

$8\sin^2\theta$

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta \, d\theta$$

$$8\sin^2\theta = 8\left[\tfrac{1}{2}(1-\cos 2\theta)\right]$$

$$= 4 - 4\cos 2\theta$$

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} 3 - 4\cos 2\theta - 2\sin\theta \, d\theta$$

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} 3-4\cos 2\theta-2\sin\theta \, d\theta$$

$$\frac{1}{2}\Big[3\theta-2\sin 2\theta+2\cos\theta\Big]_{\pi/6}^{5\pi/6}$$

$$\frac{1}{2}\left[\frac{5\pi}{2}-\overset{\sqrt{3}}{2\left(-\frac{\sqrt{3}}{2}\right)}+\overset{-\sqrt{3}}{2\left(-\frac{\sqrt{3}}{2}\right)}-\frac{\pi}{2}+\overset{\sqrt{3}}{2\left(\frac{\sqrt{3}}{2}\right)}-\overset{-\sqrt{3}}{2\left(\frac{\sqrt{3}}{2}\right)}\right]$$

$$\frac{1}{2}\big[2\pi\big] = \boxed{\pi}$$

Example 4

Find the length of the cardioid $r = 1 + \sin\theta$.

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r' = \cos\theta$$

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}$$

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$1 + 2\sin\theta + \underbrace{\boxed{\sin^2\theta + \cos^2\theta}}_{1}$$

$$\int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = 8$$

Example 5

(a) For the cardioid $r = 1 + \sin\theta$ of Example 4, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$r = 1 + \sin\theta \qquad r' = \cos\theta$

$$\frac{dy}{dx} = \frac{(\cos\theta)\sin\theta + (1+\sin\theta)\cos\theta}{(\cos\theta)(\cos\theta) - (1+\sin\theta)\sin\theta}$$

$$\frac{(\cos\theta)\sin\theta + (1+\sin\theta)\cos\theta}{(\cos\theta)(\cos\theta) - (1+\sin\theta)\sin\theta}$$

$$\boxed{\begin{gathered}\cos\theta\sin\theta + \cos\theta + \cos\theta\sin\theta \\ \cos\theta + 2\cos\theta\sin\theta \\ \cos\theta(1+2\sin\theta)\end{gathered}}$$

Top

$$\cos^2\theta - \sin\theta - \sin^2\theta$$

$$1-\sin^2\theta - \sin\theta - \sin^2\theta$$

$$1 - \sin\theta - 2\sin^2\theta$$

Bottom

$$\frac{(\cos\theta)\sin\theta + (1+\sin\theta)\cos\theta}{(\cos\theta)(\cos\theta) - (1+\sin\theta)\sin\theta} =$$

$$\frac{\cos\theta\,(1+2\sin\theta)}{1-\sin\theta-2\sin^2\theta} = \frac{dy}{dx}$$