Infinite Sequences

An **infinite sequence**, or just a **sequence**, can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the n *th term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}

a_1 first a_2 second

a_{n+1} a_n

$$\left\{\frac{1}{2^n}\right\} \qquad a_n = \frac{1}{2^n} \qquad \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, \frac{1}{2^n}, \ldots\right\}$$

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$$

If we let a_n be the digit in the n th decimal place of the number e , then $\{a_n\}$ is a sequence whose first few terms are

$$\{7,\ 1,\ 8,\ 2,\ 8,\ 1,\ 8,\ 2,\ 8,\ 4,\ 5, \ldots\}$$

The Fibonacci sequence $\{f_n\}$ is defined *recursively* by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

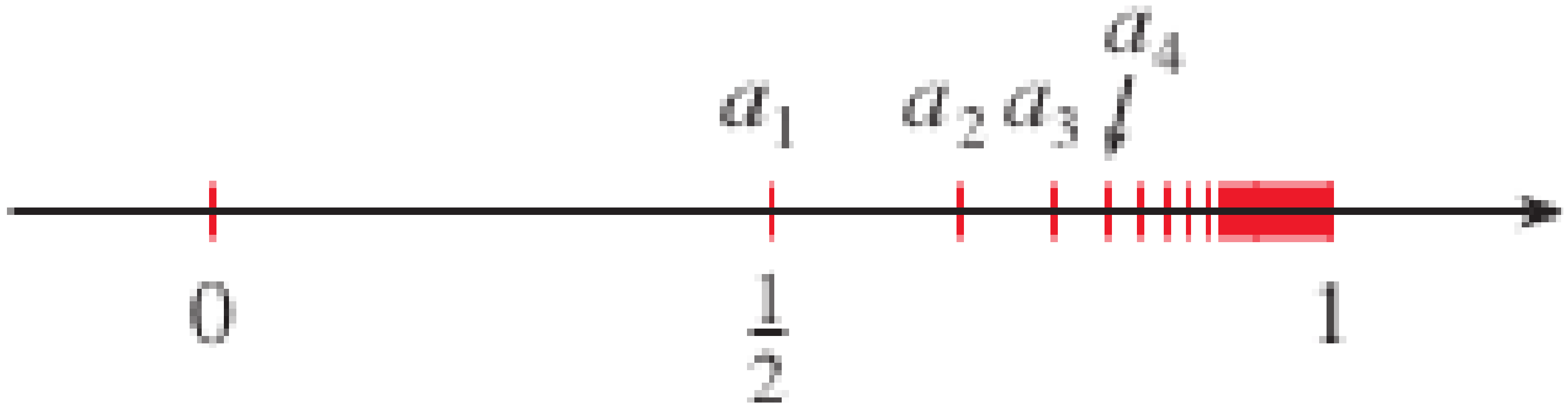
$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots\}$$

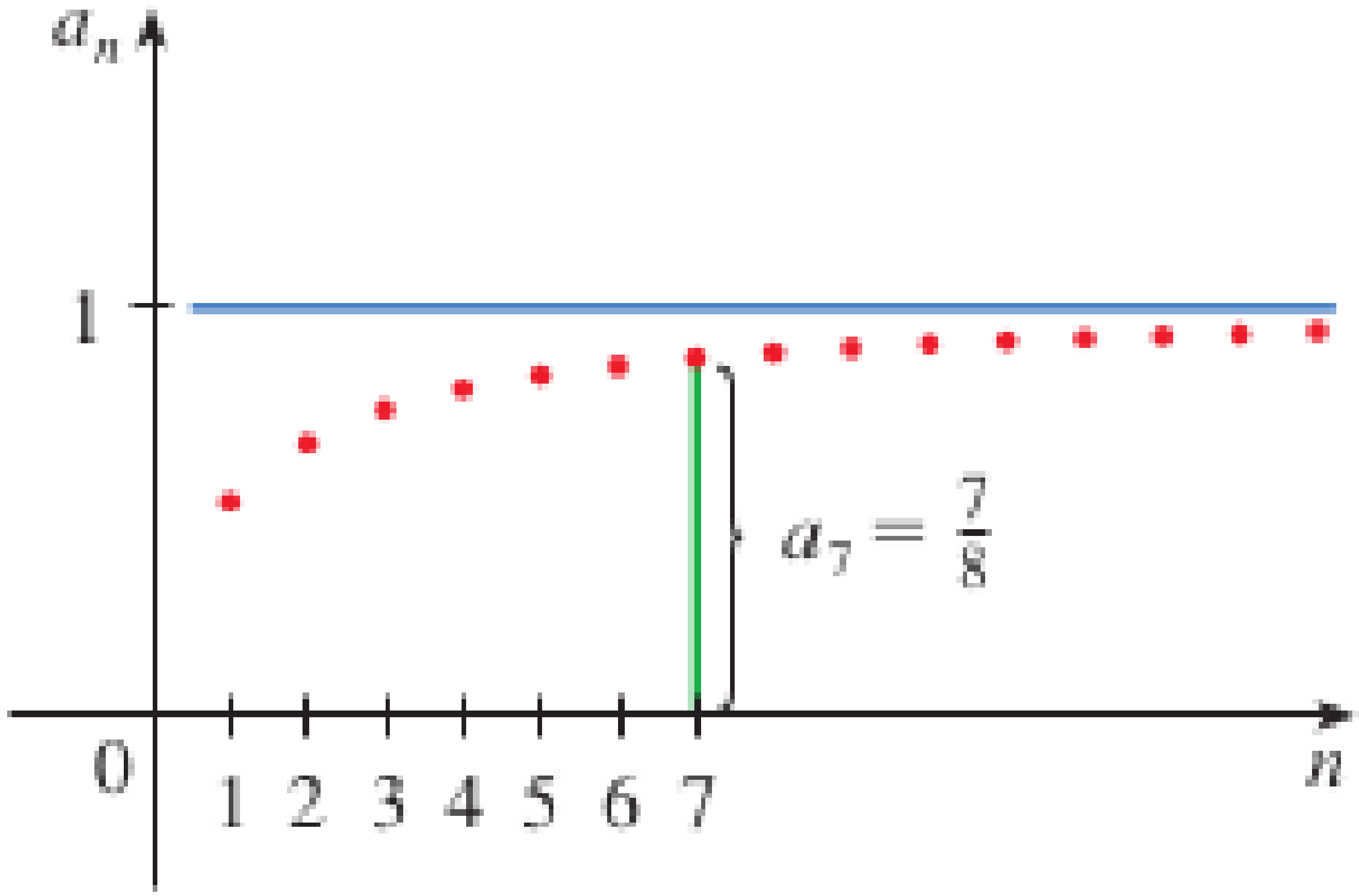
$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \cdots\right\}_{n=1}^{\infty}$$

$$\left\{(-1)^{n+1}\left(\frac{n+2}{5^n}\right)\right\}_{n=1}^{\infty}$$

$$\left\{(-1)^{n}\left(\frac{n+3}{5^{n+1}}\right)\right\}_{n=0}^{\infty}$$

$$\left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$$





A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

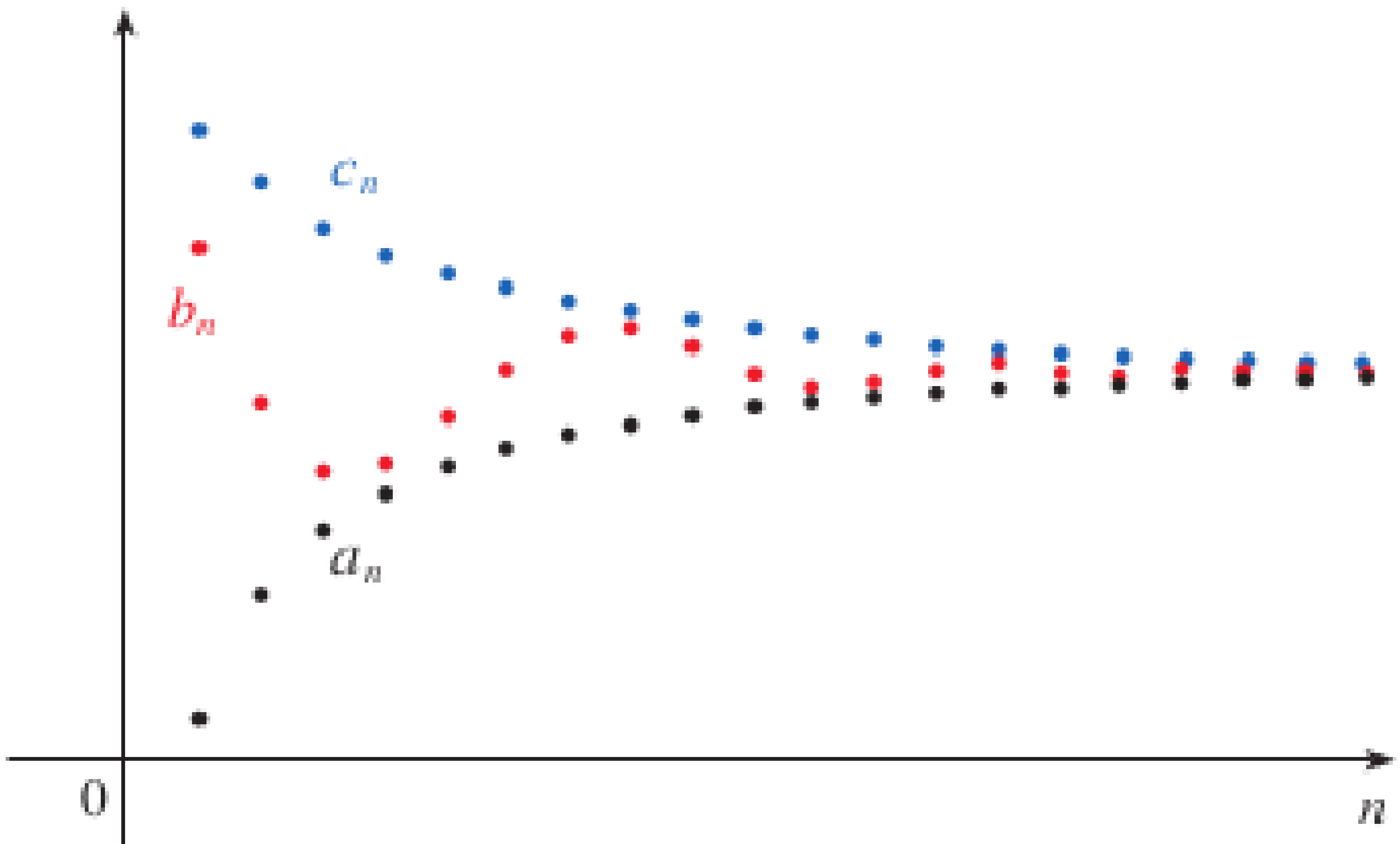
4. PRODUCT LAW $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

5. QUOTIENT LAW $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

POWER LAW $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

SQUEEZE THEOREM FOR SEQUENCES If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Figure 9



Find $\lim\limits_{n\to\infty} \dfrac{n}{n+1}$.

$$\lim_{n\to\infty} \frac{n/n}{n/n+1/n} \Rightarrow \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$\boxed{\frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}} = \frac{1}{1+0} =}$$

Is the sequence $a_n = \dfrac{n}{\dfrac{\sqrt{10+n}}{n}}$ convergent or divergent?

$$\lim_{n \to \infty} \frac{n/n}{\sqrt{\frac{10}{n^2}+\frac{n}{n^2}}}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{\frac{10}{n^2}+\frac{1}{n}}} \longrightarrow \frac{1}{\to 0} = \infty$$

Divergent

$$a_n = \frac{4n^2 - 3n}{2n^2 + 1} \Big/ n^2 \qquad \lim_{n\to\infty} \frac{4 - 3/n}{2 + 1/n^2} = \frac{4}{2} = 2$$

Convergent

$$a_n = 3^n 7^{-n} = \frac{3^n}{7^n} \qquad \lim_{n\to\infty} \frac{3^n}{7^n} = 0$$

$$= \left(\frac{3}{7}\right)^n$$

Convergent

Calculate $\lim_{n \to \infty} \frac{\ln n}{n}$.

$$\lim_{x \to \infty} \frac{\ln|x|}{x} \to \frac{1/x}{1}$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{n \to \infty} \frac{\ln n}{n} \to 0$$

Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Divergent

If

$$\lim_{n\to\infty} |a_n| = 0$$

then

$$\lim_{n\to\infty} a_n = 0$$

Evaluate $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|$ if it exists.

$$\lim_{n\to\infty} \frac{1}{n} \to 0$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n} \Rightarrow 0$$

Find $\lim_{n\to\infty} \sin\frac{\pi}{n}$. $\rightarrow$ $\sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

continuous

$\sin(0) = \boxed{0}$

$$\lim_{n\to\infty} \sin\left(\frac{n}{n+1}\right)$$

$$\sin\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \boxed{\sin(1)}$$

Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

$$\lim_{n \to \infty} \frac{n!}{n^n}$$

$n = 1 \quad \frac{1}{1}$

$n = 2 = \frac{2 \cdot 1}{2 \cdot 2} = \frac{1}{2}$

$$a_n = \frac{1}{n} \left(\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n} \right)$$

$n = 3 = \frac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3} = \frac{1}{3} \cdot \frac{2}{3}$

$n = 4 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 4 \cdot 4 \cdot 4} = \frac{1}{4} \cdot \frac{2}{4} \cdot \frac{3}{4}$

$$a_n = \frac{n!}{n^n}$$

$$0 < \boxed{a_n} < \frac{1}{n}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0$$

For what values of r is the sequence $\{r^n\}$ convergent?

r^n

$(-1)^n$

$-1 < r \leq 1$ convergent

$r \leq -1$

$r > 1$ divergent

$-1 < r < 1$ $\quad 0 \leq |r| < 1$ convergent

$r = \cancel{-1}, 0, 1$ $\quad |r| > 1$ divergent

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$.

$$\frac{n}{n^2+1} > \frac{(n+1)}{(n+1)^2+1}$$

A sequence is called **monotonic** if it is either increasing or decreasing.

Example 13

Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$\frac{1-x^2}{(x^2+1)^2} < 0$$

$$1-x^2 < 0$$
$$-x^2 < -1$$
$$x^2 > 1$$

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

A sequence is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

Every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.