1 $$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R$$

2 $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R$$

3 $$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots \qquad |x-a| < R$$

4 $$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4\boxed{(x-a)} + \boxed{3 \cdot 4 \cdot 5c_5}(x-a)^2 + \cdots \qquad |x-a| < R$$

$$C = n!\, c_n \qquad \frac{f^{(n)}(a)}{n!} = c_n$$

5 **Theorem** If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

6
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

7
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Maclaurin

EXAMPLE 2 For the function $f(x) = e^x$, find the Maclaurin series and its radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$x=0$$

$$f(x) = e^x \rightarrow 1$$

$$f'(x) = e^x \rightarrow 1$$

$$f''(x) = e^x \rightarrow 1$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

Ratio Test

< 1 convergent

> 1 divergent

$= 1$ Don't know

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}$$

$$= \frac{x \cdot \cancel{x^n}}{\cancel{x^n}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{(n!)}} = \frac{x}{n+1} \rightarrow 0 < 1$$

$$(-\infty, \infty)$$

EXAMPLE 6 Find the Maclaurin series for cos x.

$x=0$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$\cos(x) \to 1 \quad 0$

$\downarrow$

$-\sin(x) \to 0$

$-\cos(x) \to -1$

$\sin(x) \to 0$

$\cos(x) \to 1$

$-\sin(x) \to 0$

$-\cos(x) \to -1$

$$f(x)=1^0+\cancel{\frac{0}{1!}x^1}+\frac{-1}{2!}x^{2}+\cancel{\frac{0}{3!}x^3}+\frac{1}{4!}x^{4}+\cancel{\frac{0}{5!}x^5}\ldots$$

$$f(x)=\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

$$f(x) = x\cos x$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n!)}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$f(x) = x\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1}$$

$$f(x) = \ln(1 + 3x^2) \qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(3x^2)^n}{n}$$

$2n$

$$\boxed{f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} (3)^n \frac{x^{2n}}{n}}$$

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

$$u = -x^2$$

$$\int e^{-x^2}$$

$$du = -2x\,dx$$

$$-\frac{du}{2x} = dx$$

$$\int \frac{-1}{2x} e^{u}\,du$$

$$\int x e^{-x^2}$$

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$\int e^{-x^2} = \int \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} \cdots\right)$$

$$= \frac{x^1}{1 \cdot 0!} - \frac{x^3}{3 \cdot 1} + \frac{x^5}{5 \cdot 2} - \frac{x^7}{7 \cdot 6} + \frac{x^9}{9 \cdot 24} - \frac{x^{11}}{11 \cdot 120} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}$$

$0! = 1$ $3! = 6$

$1! = 1$ $4! = 24$

$2! = 2$ $5! = 120$

EXAMPLE 1

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leqslant x \leqslant 9$?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$a = 8$

$f^{(0)}(8) = \sqrt[3]{8} = 2$

$f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{12}$

$f''(8) = \frac{-2}{9(8)^{5/3}} = \frac{-2}{288} = \frac{-1}{144}$

$x^{1/3} \to \frac{1}{3} x^{-2/3} = \frac{1}{3 \cdot x^{2/3}}$

$f(x) = T_2(x)$

$$= \frac{2}{0!}(x-8)^0 + \frac{1/12}{1!}(x-8)^1 + \frac{-1/144}{2!}(x-8)^2$$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 = \sqrt[3]{x}$$

9 Taylor's Inequality If $\boxed{|f^{(n+1)}(x)| \leqslant M}$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

$$f^{(3)}(x) = -\frac{2}{9}\left(\frac{-5}{3}\right)x^{-8/3} = \frac{10}{27}x^{-8/3}$$

$$f^{(2)}(x) = -\frac{2}{9}x^{-5/3}$$

$$\frac{10}{27}\cdot\frac{1}{8^{8/3}} \leq \frac{10}{27}\cdot\frac{1}{7^{8/3}} < 0.0021$$

$$|R_2(x)| \leq \frac{0.0021}{(2+1)!}|7-8|^{2+1}$$

$$\leq \frac{0.0021}{6} \approx 0.0004$$

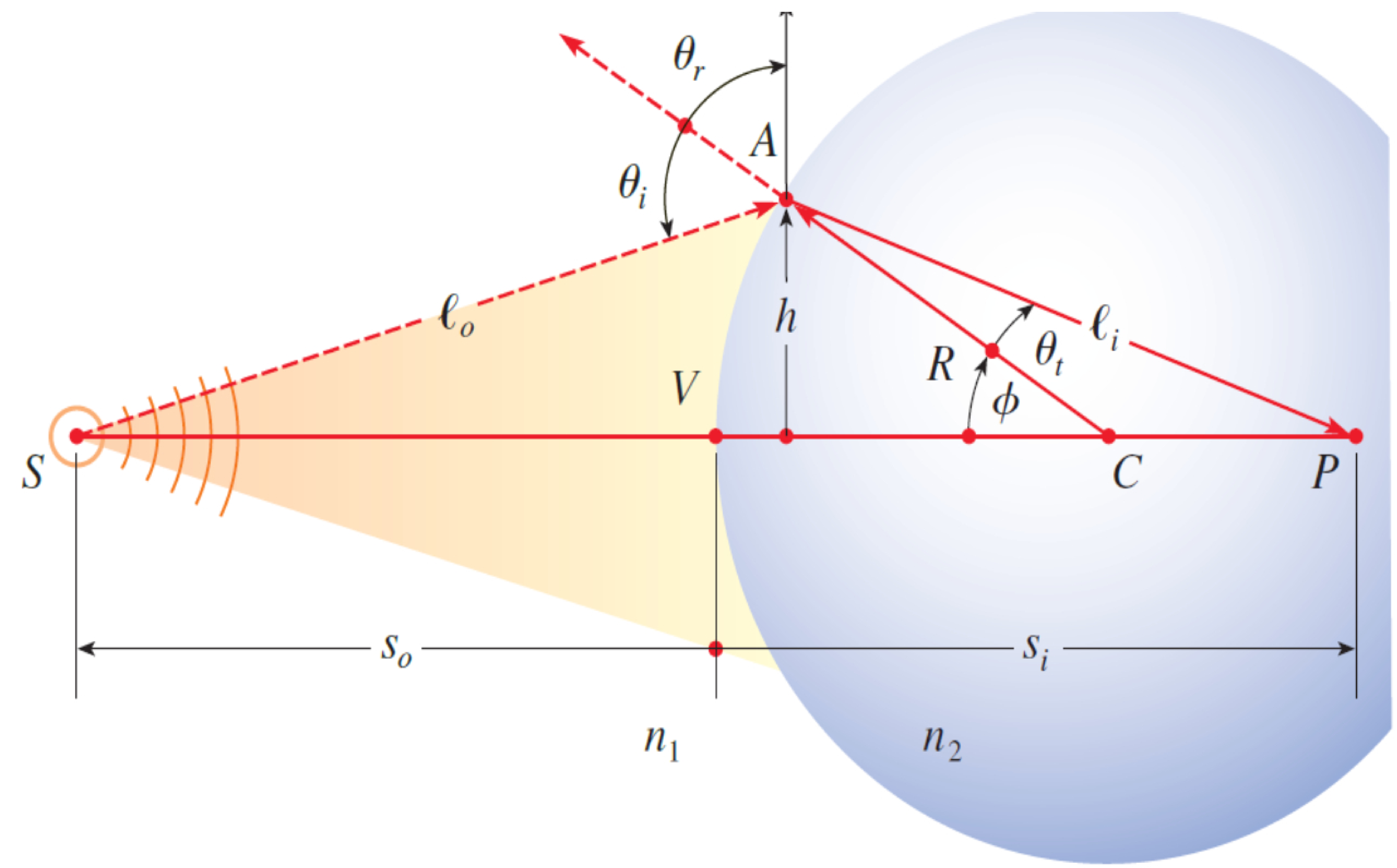

FIGURE 8
Refraction at a spherical interface

If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the earth.

(a) If R is the radius of the earth and L is the length of the highway, show that the correction is

$$C = R\sec(L/R) - R$$

(b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}$$

the corrections given by the formulas in
and (b) for a highway that is 100 km long.
radius of the earth to be 6370 km.)