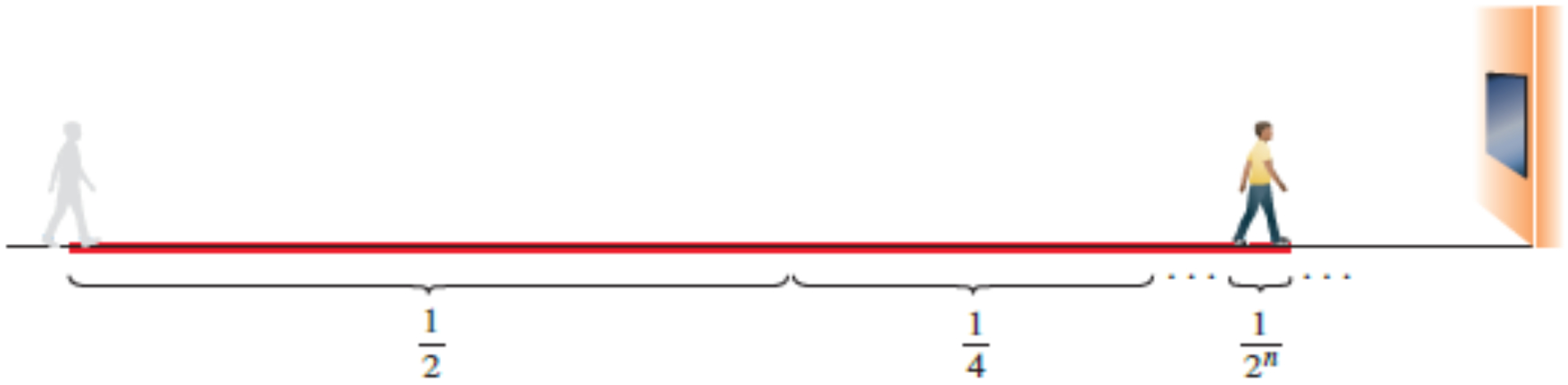
Infinite Series

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1}$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$



2 **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$S_3 = a_1 + a_2 + a_3$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \boxed{\frac{2n}{3n+5}} \qquad S = \frac{2}{3}$$

$$\lim_{n\to\infty} \frac{2n \,/n}{3n+5\,/n} \to \frac{2}{3}$$

EXAMPLE 2 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \qquad \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$1 = A(n+1) + Bn$$

$n=0 \quad 1 = A$

$n=-1 \quad 1 = -B \rightarrow -1 = B$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1} \rightarrow S = 1$$

$$\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right) \cancel{}$$

$$\ldots \cancel{\frac{1}{n}} \left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} 1-\frac{1}{n+1} \rightarrow 1$$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \boxed{\frac{a}{1-r}} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

$n=1$

$r =$ ratio

$a = 5$

$r = -\frac{2}{3}$

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{5/3} = 3$$

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975

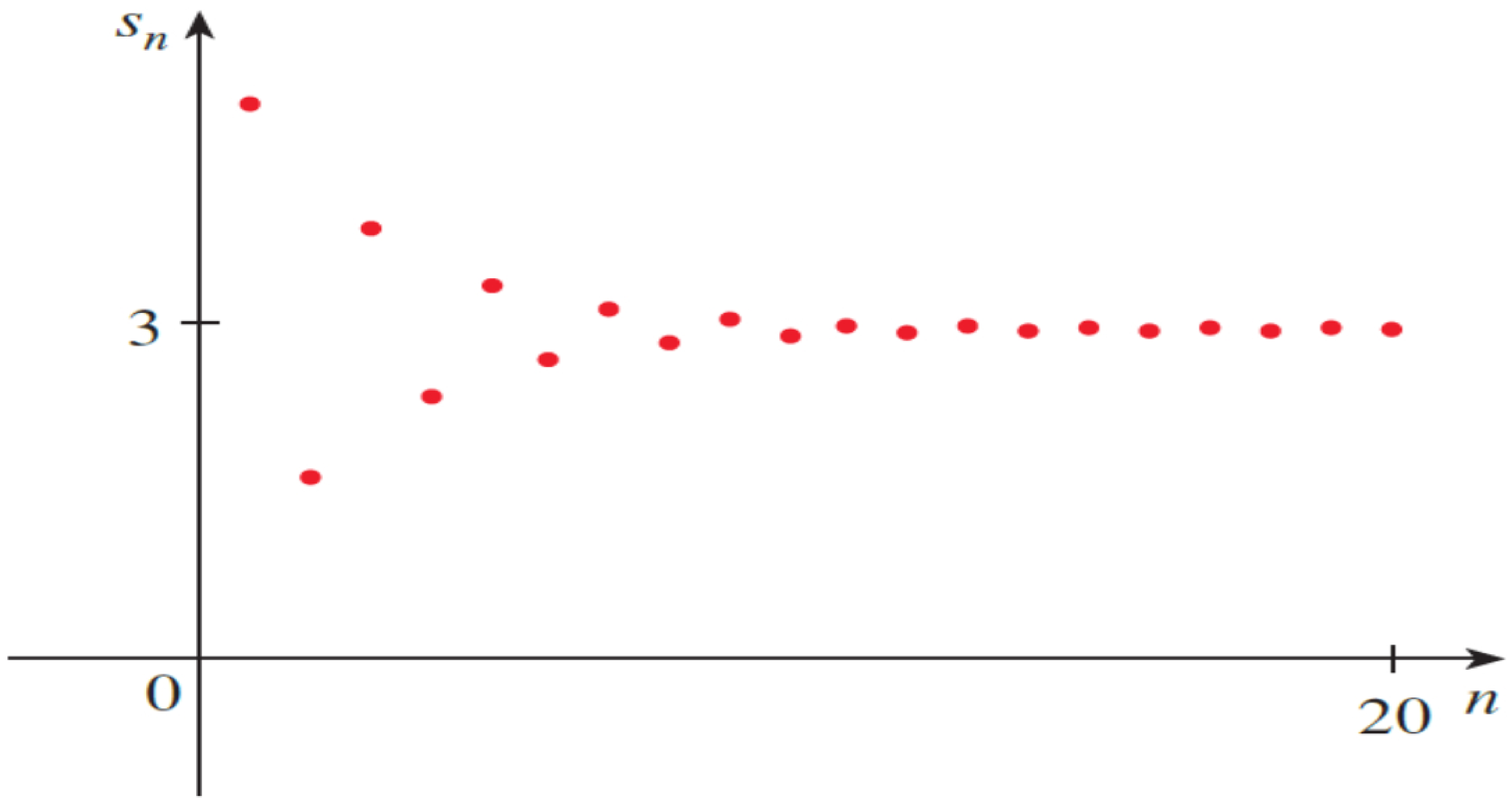


FIGURE 4

EXAMPLE 4 Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

$$\sum_{n=1}^{\infty} ar^{\boxed{n-1}}$$

$$(2^2)^n (3)(3)^{-n}$$

$$r = \frac{4}{3}$$

$$\sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

$$\frac{(4)^n (3)}{3^n} = (3)\left(\frac{4}{3}\right)^n = (3)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)^{n-1}$$

$$2 \cdot 2 \cdot 2 = 2^3 \rightarrow 2^n$$

$$= 2 \cdot 2^2 \rightarrow 2 \cdot 2^{n-1}$$

day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration just after the third injection.

(b) What is the concentration just after the nth dose?

(c) What is the limiting concentration?

$C_0 = 0$

$C_1 = 0.2 + (.30)(0) = 0.2$

$C_2 = 0.2 + (.30)(0.2) = 0.26$

$C_3 = 0.2 + (.30)(0.26) = 0.278$

C_3

$C_1 = 0.2 + (.30)(0)$

$C_2 = 0.2 + (.30)(.2 + .30(0))$

$C_3 = 0.2 + .30(0.2 + (.30)(.2 + .30(0)))$

$\vdots$

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \cdots + 0.2(0.3)^{n-1}$$

$a = 0.2 \qquad r = .3$

$S = \dfrac{0.2}{1-.3} = \dfrac{0.2}{0.7} = \boxed{\dfrac{2}{7}}$

0.28571428571

$$2.3 + \Big[\frac{\quad}{10^3} + \frac{\quad}{10^5} + \frac{\quad}{10^7} + \frac{\quad}{10^{\,}}$$

$$3 + \sum_{n=1}^{\infty}\left(\frac{17}{10^3}\right)\left(\frac{1}{10^2}\right)^{n-1} = 2.3 + \frac{17/10^3}{1 - 1/10^2}$$

$$\frac{23}{10} + \left(\frac{17}{10^3}\right)\left(\frac{10^2}{99}\right)$$

$$\frac{23}{10} + \frac{17}{990} = \frac{2294}{990} = \boxed{\frac{1147}{495}}$$

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

7 Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

EXAMPLE 9 Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$a_n = \frac{n^2}{5n^2+4}$$

$$\lim_{n\to\infty} \frac{n^2 / n^2}{5n^2+4 / n^2}$$

$$\lim_{n\to\infty} \frac{1}{5+4/n^2} \to \frac{1}{5} \neq 0$$

Diverges

Test for Divergence

EXAMPLE 8 Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$S_1 = 1$

$S_2 = \frac{3}{2}$

$S_4 > 2$

$S_8 > \frac{5}{2}$

$S_{16} > 3$

$S_{32} > \frac{7}{2}$

$S_{64} > 4$

$$s_2 = 1 + \tfrac{1}{2} = \frac{3}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2} > 2$$

$$s_8 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) > \frac{5}{2}$$

$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right)$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}$$

Properties of Convergent Series

The following properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1.

8 Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

EXAMPLE 10 Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$. $=\boxed{4}$

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} \rightarrow 3(1) = 3$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \rightarrow \frac{1/2}{1-1/2} = 1$$

43. $\sum_{k=1}^{\infty} (\sin 100)^k$

$(\sin 100)(\sin 100)^{k-1}$

a $\quad r$

$$\boxed{\frac{\sin 100}{1 - \sin 100}}$$