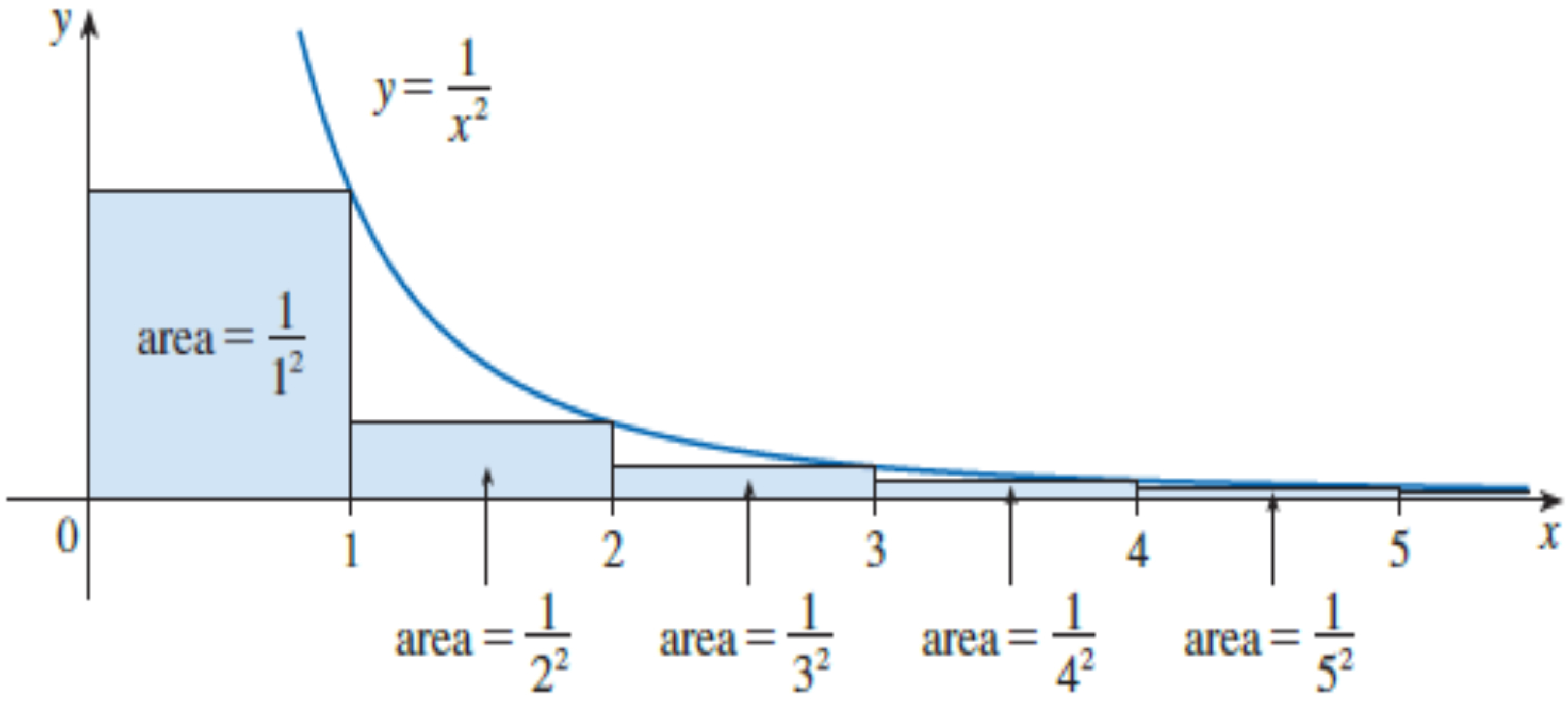
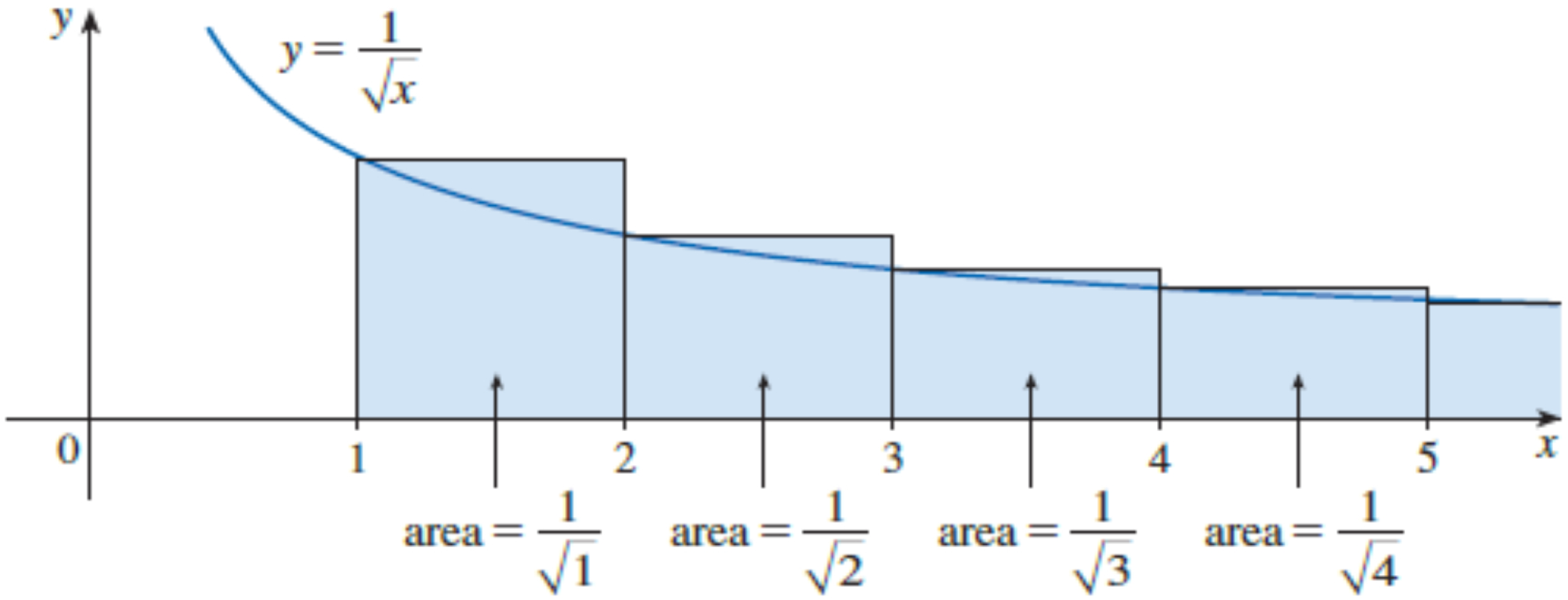
The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.


y
$y = \frac{1}{x^2}$
area = $\frac{1}{1^2}$
0
1
2
3
4
5
x
area = $\frac{1}{2^2}$
area = $\frac{1}{3^2}$
area = $\frac{1}{4^2}$
area = $\frac{1}{5^2}$


y
$y = \frac{1}{\sqrt{x}}$
0
1
2
3
4
5
x
area = $\frac{1}{\sqrt{1}}$
area = $\frac{1}{\sqrt{2}}$
area = $\frac{1}{\sqrt{3}}$
area = $\frac{1}{\sqrt{4}}$

EXAMPLE 1 Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

1) continuos ✓ $n^2+1=0 \rightarrow n=\sqrt{-1}$

2) positive ✓

3) decreasing ✓

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx \rightarrow \tan^{-1}(x)\Big]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Convegent

EXAMPLE 2 For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

$p < 0$ $\quad \sum_{n=1}^{\infty} n^p$ $\quad$ Test for divergence

(Diverges)

$p = 0$ $\quad \sum_{n=1}^{\infty} 1$ $\quad$ Test for divergence

(Diverges)

$p > 0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad \int_1^{\infty} \frac{1}{x^p}\,dx$

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$$

$$\left.\frac{x^{-p+1}}{-p+1}\right]_1^\infty = \frac{1}{-p+1}\left(x^{-p+1}\right)_1^\infty$$

$$= \frac{1}{-p+1}\left(\frac{1}{x^{p-1}}\right)_1^\infty$$

$$= \frac{1}{-p+1}(-1) = \boxed{\frac{1}{p-1}}$$

$p = 1$ divergent

$p > 1$ Convergent

$0 < p < 1$ Diverget

$p > 1$ convergent

$p < 1$ Divergent

$\boxed{1}$ The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

EXAMPLE 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

$n > 1$ continous and positive

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) \qquad \frac{f'g - fg'}{g^2} \qquad \begin{aligned} f &= \ln x \\ f' &= 1/x \\ g &= x \\ g' &= 1 \end{aligned}$$

$$\frac{(1/x)x - (\ln x)(1)}{x^2} = \boxed{\frac{1-\ln x}{x^2}}$$

$$\frac{1-\ln x}{x^2} < 0 \quad \rightarrow \quad 1 - \ln x < 0$$

$$-\ln x < -1$$

$$\int_1^\infty \frac{\ln x}{x}\,dx$$

$u = \ln x$

$du = 1/x\,dx$

$u(1) = 0$

$u(\infty) = \infty$

$$\int_0^\infty u\,du = \frac{u^2}{2}\Big]_0^\infty$$

$\rightarrow \boxed{\infty}$

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is Divergent

$\boxed{2}$ **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

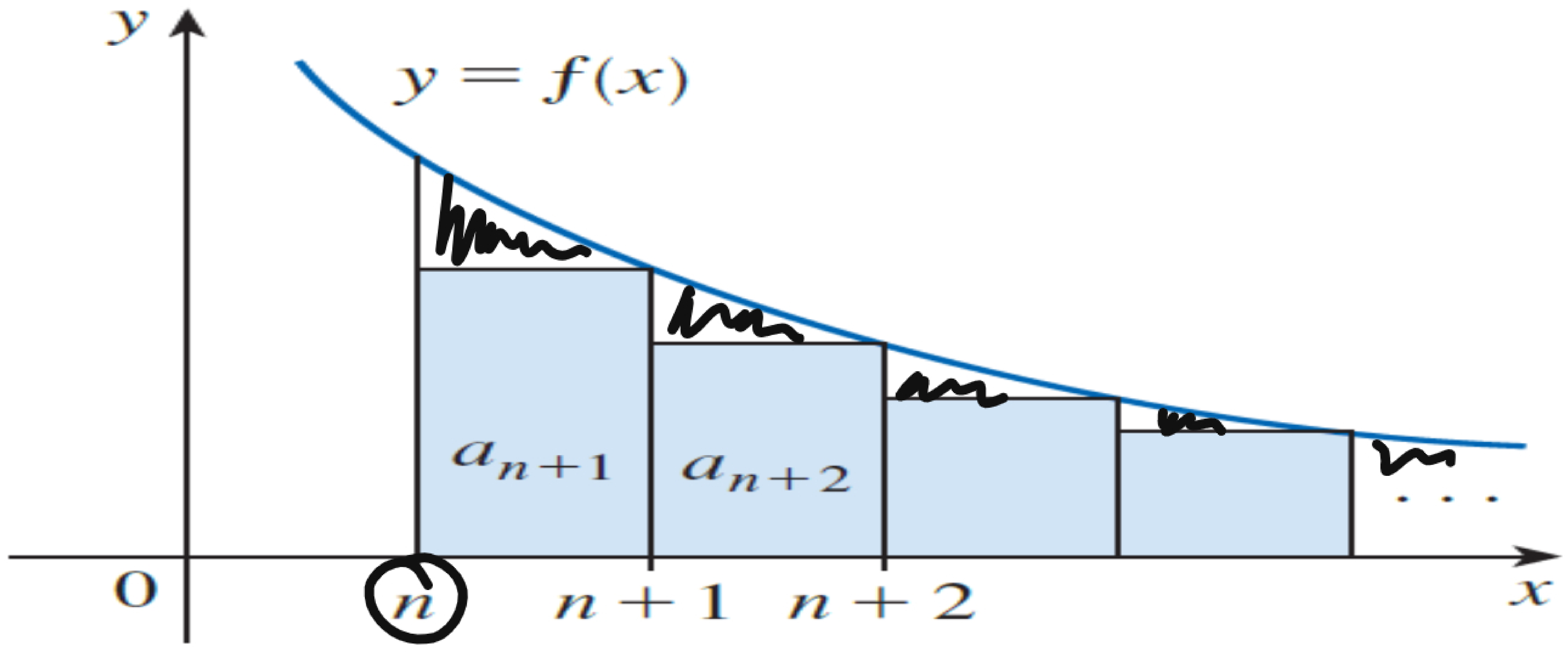


FIGURE 3

BitPaper Share Knowledge

EXAMPLE 5

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

a) $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \frac{1}{10^3}$

$$\sum_{n=1}^{10} \frac{1}{n^3}$$

$$= 1.19753198567$$

$$R_n \le \frac{1}{200}$$

$.005$

$$\int_{10}^{\infty} \frac{1}{x^3} = \frac{-x^{-2}}{2}\Big]_{10}^{\infty}$$

(x^{-3})

$$= -\frac{1}{2x^2}\Big]_{10}^{\infty} = \left(0 - \left(\frac{-1}{200}\right)\right)$$

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

$$\int_n^{\infty} \frac{1}{x^3} = -\frac{1}{2x^2}\Big]_n^{\infty} = \frac{1}{2n^2} \le 0.0005$$

$$1 \le 0.0005(2n^2)$$

$$1 \le .001n^2$$

$$\sqrt{1000} \le \sqrt{n^2}$$

$$31.\ldots \le n$$

$$\boxed{32 = n}$$

$$\sum_{n=1}^{10} \frac{1}{n^3}$$

$$= 1.19753198567$$

$$R_n \le \frac{1}{200}$$

$$.005$$

$S_{10} = 1.19753198567$

PLE 6 Use (3) with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

$$S_{10} + \int_{11}^{\infty} \frac{1}{x^3}\,dx \le S \le S_{10} + \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

$$\frac{1}{2n^2}$$

$+ 1/242$ $\qquad$ $+ 1/200$

$$1.19753198567 + \frac{1}{200}$$

$$1.19753198567 + \frac{1}{242}$$

$$\sum_{n=1}^{10\,000\,000\,000} \frac{1}{n^3} = 1.20205690315$$

$$\frac{1.201664217 + 1.202531986}{2} = 1.202098102$$

$n=1 \quad n^{\sqrt{2}}$ $\qquad n=3$

13. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

16. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdot$ $\quad \frac{1}{n\sqrt{n}}$

C

$p = \frac{3}{2} > 1$

$\frac{1}{n \cdot n^{1/2}} \quad \frac{1}{n^{3/2}}$