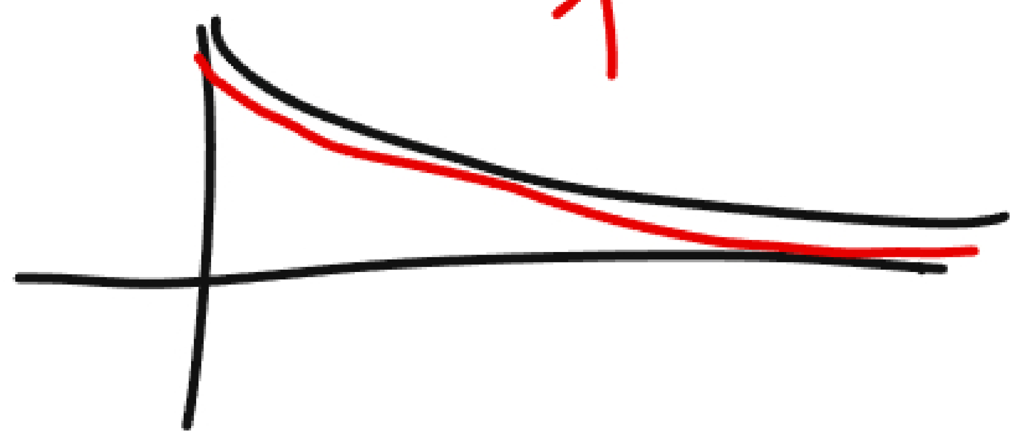
$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$ Convergent

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2+4n} < \frac{5}{2n^2} = \frac{5}{2}\,\frac{1}{n^{2}}$$

P-series

$p = 2 > 1$

Converges



- A p-series [$\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1$; see (11.3.1)]
- A geometric series [$\Sigma\, ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1$; see (11.2.4)]

The Direct Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \quad \text{divergent}$$

$$\frac{\ln(n)}{n} > \frac{1}{n} \qquad n \geq 3 \qquad P$$

$p = 1 \not> 1$

divergent

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$\frac{1}{2^n - 1} > \underbrace{\frac{1}{2^n}}_{C} = \left(\frac{1}{2}\right)^n \qquad r = \frac{1}{2}$$

$$\frac{1}{2} < 1$$

convergent

The Limit Comparison Test Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

$a_n = \dfrac{1}{2^n - 1}$ (C) $b_n = \dfrac{1}{2^n}$ (C)

$$\lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \frac{2^n}{2^n-1} = \frac{2^n/2^n}{2^n/2^n - 1/2^n} = \frac{1}{1-1/2^n} \to 1$$

$$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$$ (C) Direct Comparison to $\left(\frac{9}{10}\right)^n$

$$\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$$

(C) $9/10 < 1$

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$$ Diverges L.C.T.

$$\frac{2}{\sqrt{n}+2} < \frac{2}{\sqrt{n}} = \frac{2}{n^{1/2}} \quad p = 1/2 \le 1$$

Diverges

$$\frac{\frac{2}{\sqrt{n}+2}}{\frac{2}{\sqrt{n}}} = \frac{\sqrt{n}/\sqrt{n}}{\sqrt{n}+2/\sqrt{n}} = \frac{1}{1+2/\sqrt{n}} \to 1$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies the conditions

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

$$\boxed{\lim_{n \to \infty} b_n \neq 0}$$

Test for Divergence

The alternating harmonic series Convergent A.S.T.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = (-1)^{n-1}\left(\frac{1}{n}\right)$$

1) $b_{n+1} \le b_n \to \frac{1}{n+1} \le \frac{1}{n}$ ✓

2) $\lim_{n\to\infty} b_n = 0 \to \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$$\sum_{n=1}^{\infty} \frac{\boxed{(-1)^n} 3n}{4n-1}$$

$$b_n = \frac{3n}{4n-1}$$

1) $\frac{3(n+1)}{4(n+1)-1} < \frac{3n}{4n-1}$

2) $\lim_{n\to\infty} \frac{3n}{4n-1} \frac{/n}{/n} = \frac{3n/n}{4n/n - 1/n} \to \frac{3}{4} \neq 0$

Divergent

Test for Divergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \quad (C)$$

$$b_n = \frac{1}{n^2}$$

1) $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ ✓

2) $\lim_{n \to \infty} \frac{1}{n^2} = 0$ ✓

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad (c)$$

$$p = 2 > 1$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} \dots$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$$

$\boxed{1}$ **Definition** A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

$\boxed{2}$ **Definition** A series Σa_n is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, if Σa_n converges but $\Sigma |a_n|$ diverges.

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$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1} + \frac{\cos 2}{4} + \cdots\cdots +$$

(C)

Absolutly convergent $\frac{\cos 91}{91^2}$

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$$

$$\frac{|\cos n|}{n^2} \le \frac{1}{n^2} \quad p=2>1$$

(C) (C)

The Ratio Test

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n.

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

$$a_n = \frac{n^3}{3^n} \qquad a_{n+1} = \frac{(n+1)^3}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} \right| = \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right|$$

$$= \left| \frac{3^n}{3^{n+1}} \cdot \frac{(n+1)^3}{n^3} \right|$$

$3^{n+1} =$

$3(3^n)$

$$\lim_{n\to\infty} \frac{1}{3} \cdot \left(\frac{n+1}{n}\right)^3 \rightarrow$$

$$\lim_{n\to\infty} \frac{1}{3} \left(1+\frac{1}{n}\right)^3 \rightarrow \boxed{\frac{1}{3}}$$

$$a_n = \frac{n^n}{n!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$\frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)(n+1)^n}{n^n}$$

$$\lim_{n\to\infty} \left| \boxed{\frac{(n+1)^{n+1}}{n^n}} \cdot \boxed{\frac{n!}{(n+1)!}} \right|$$

$$(n+1)\left(\frac{n+1}{n}\right)^n$$

$$(n+1)\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} (n+1)\left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n+1}$$

$$\frac{n!}{(n+1)(n)!} = \frac{1}{n+1}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \to e$$

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^{n}$$

$$\sqrt[n]{a^n} \rightarrow a^{n/n}$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}$$

$$\downarrow$$

$$\lim_{n\to\infty} \frac{2n+3/n}{3n+2/n} \rightarrow \frac{2}{3}$$

Determine whether the series $\sum_{n=1}^{\infty} \boxed{\left(\frac{n}{n+1}\right)^n}$ converges or diverges.

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{n+1}} \rightarrow \lim_{n\to\infty} \frac{n\;/n}{n+1\;/n} = 1$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \boxed{\frac{1}{e}}$$

$$\left(\frac{n}{n} + \frac{1}{n}\right)^n$$

$$\left(1 + \frac{1}{n}\right)^n$$

Divergent

Test for Divergence