Power Series

A **power series** is a series of the form

1

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Geometric

$-1 < x < 1$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

EXAMPLE 1 For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

$$a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$$

$$a_n = \frac{(x-3)^n}{n}$$

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{n}{n+1} \right|$$

$(x-3)(x-3)^n$

$\frac{1}{1+1/n}$

$$= |x-3| < 1$$

$$|x-3| < 1$$

$$-1 < x-3 < 1$$

$$+3 \quad +3 \quad +3$$

$$\boxed{2 < x < 4}$$

$x = 2 \quad \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \frac{(-1)^n}{n}$ Converges

$x = 4 \quad \sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \frac{1}{n}$ Diverges

$$\boxed{2 \le x < 4}$$

EXAMPLE 2 For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} \left| (n+1)(x) \right|$$

$(n+1)!$

$(n+1)(n!)$

$x^{n+1} \to x^n \cdot x^1$

$$\lim_{n\to\infty} (n+1)|x| = \infty$$

Diverges $x \neq 0$

EXAMPLE 3 For what values of x does the series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converge?

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(2(n+1))!}{x^n/(2n)!} \right| = \lim_{n\to\infty} \left| \frac{x}{(2n+2)(2n+1)} \right| = 0 < 1$$

$$\frac{x^{n+1}}{x^n} = \frac{x^n \cdot x}{x^n} = |x|$$

Converges

$x = \mathbb{R}$

$$\frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)}$$

4 Theorem For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 2	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Representations of Functions using Geometric Series

We will obtain power series representations for several functions by manipulating geometric series. We start with an equation that we have seen before.

1

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

EXAMPLE 1 Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

EXAMPLE 2 Find a power series representation for $1/(x + 2)$.

EXAMPLE 3 Find a power series representation of $x^3/(x + 2)$.

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

EXAMPLE 4 Express $1/(1 - x)^2$ as a power series by differentiating Equation 1. What is the radius of convergence?

EXAMPLE 5 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.