BitPaper
Share Knowledge

**EXAMPLE 7** Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

**SOLUTION** Notice that this series starts with $n = 0$ and so the first term is $x^0 = 1$. (With series, we adopt the convention that $x^0 = 1$ even when $x = 0$.) Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and (4) gives

5
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

$$\boxed{\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1}$$

BitPaper
Share Knowledge

**EXAMPLE 1** Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

$$\boxed{\frac{1}{1+x^2}} = \sum_{n=0}^{\infty} (-x^2)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

$$1-(-x^2)$$

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} (u)^n \qquad u = -x^2$$

BitPaper
Share Knowledge

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$(-1, 1)$$

Interval of convergence

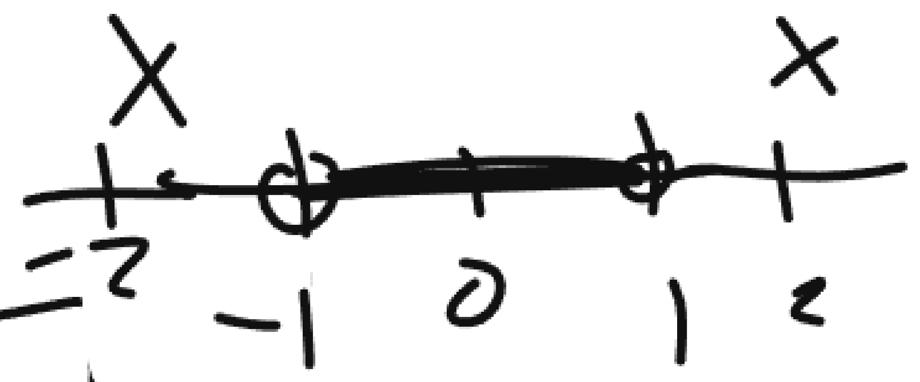
geometric series

$$|u| < 1$$

$$|-x^2| < 1$$

$$\sqrt{x^2} < \sqrt{1}$$

$$\boxed{-1 < x < 1}$$



**EXAMPLE 3** Find a power series representation of $x^3/(x+2)$.

$$\frac{x^3/2}{x+2} \rightarrow \frac{1}{1-u} = \sum_{n=0}^{\infty} x^n$$

$$\frac{2}{2} + \frac{x}{2}$$

$$1 + \frac{x}{2}$$

$$1 - \left(\frac{-x}{2}\right)$$

$$= \frac{x^3}{2} \cdot \frac{1}{1-\left(\frac{-x}{2}\right)}$$

$$\frac{x^3}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{x^3}{2^1}\right)\left(\frac{-x}{2^n}\right)^n$$

$$x^3 \cdot x^n = x^{n+3}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{2^{n+1}} = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{2^{n+1}}} x^{n+3}$$

**2 Theorem** If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function $f$ defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both $R$.

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n]$$

(iv) $$\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$$

BitPaper
Share Knowledge

BitPaper
Share Knowledge

**EXAMPLE 5** Find a power series representation for $\ln(1+x)$ and its radius of convergence.

$$\ln(1+x) \qquad\qquad \ln(x) \xrightarrow{\partial/dx} \frac{1}{x}$$

$$\int \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$$

$$\boxed{\ln(1+x)} = \int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^n \, dx$$

$$= \int 1 - x + x^2 - x^3 + x^4 + \ldots$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots$$

BitPaper
Share Knowledge

**EXAMPLE 6** Find a power series representation for $f(x) = \tan^{-1}x$.

$$f(x) = \tan^{-1} x$$

$$\boxed{\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n}$$

$$\tan^{-1}x = \int \frac{1}{1+x^2}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

BitPaper
Share Knowledge

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \longrightarrow \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

BitPaper
Share Knowledge

## EXAMPLE 7

(a) Evaluate $\int [1/(1+x^7)]\,dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} [1/(1+x^7)]\,dx$ correct to within $10^{-7}$.

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

$$\frac{1}{1+x^7} \to \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

$$\int \frac{1}{1+x^7}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

BitPaper
Share Knowledge

# EXAMPLE 7

(a) Evaluate $\int [1/(1 + x^7)]\,dx$ as a power series.

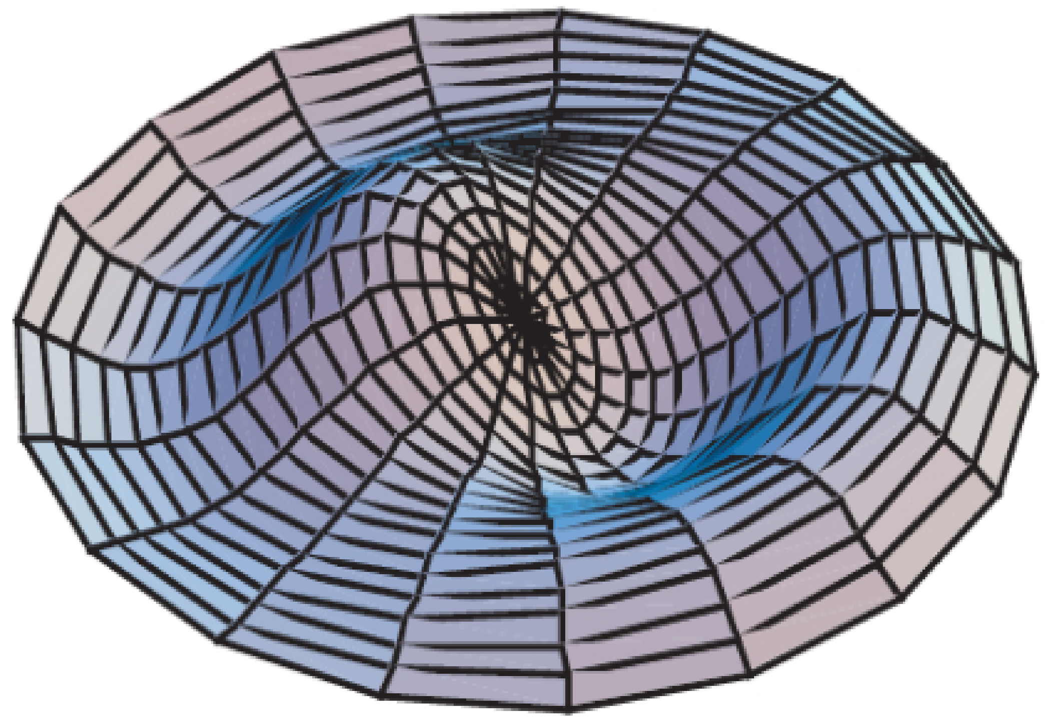
(b) Use part (a) to approximate $\int_0^{0.5} [1/(1 + x^7)]\,dx$ correct to within $10^{-7}$.

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}}\right]_{x=0}^{x=1/2}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{7n+1}}{7n+1} =$$

$$= x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \frac{x^{29}}{29}$$

BitPaper
Share Knowledge



A computer-generated model, involving Bessel functions and cosine functions, of a vibrating drumhead.

**EXAMPLE 8** The Bessel function of order 0 is defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

The first few partial sums are

$$s_0(x) = 1$$

$$s_1(x) = 1 - \frac{x^2}{4}$$

$$s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

$$s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$