The Dot Product of Two Vectors

To find the dot product of vectors $\mathbf{a}$ and $\mathbf{b}$ we multiply corresponding components and add.

1 Definition of the Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\langle 2, 4\rangle \cdot \langle 3, -1\rangle = (2)(3) + (4)(-1) = \boxed{2}$$

$$6 - 4$$

$$\langle -1, 7, 4\rangle \cdot \left\langle 6, 2, -\frac{1}{2}\right\rangle = (-1)(6) + (7)(2) + (4)\left(-\frac{1}{2}\right) = \boxed{6}$$

$$-6 + 14 - 2$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = (1)(0) + 2(2) + (-3)(-1) = \boxed{7}$$

$$0 + 4 + 3$$

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

$(3, 2, 4)\ (3, 2, 4)$

$\left(\sqrt{(3)(3)+(2)(2)+(4)(4)}\right)^2$

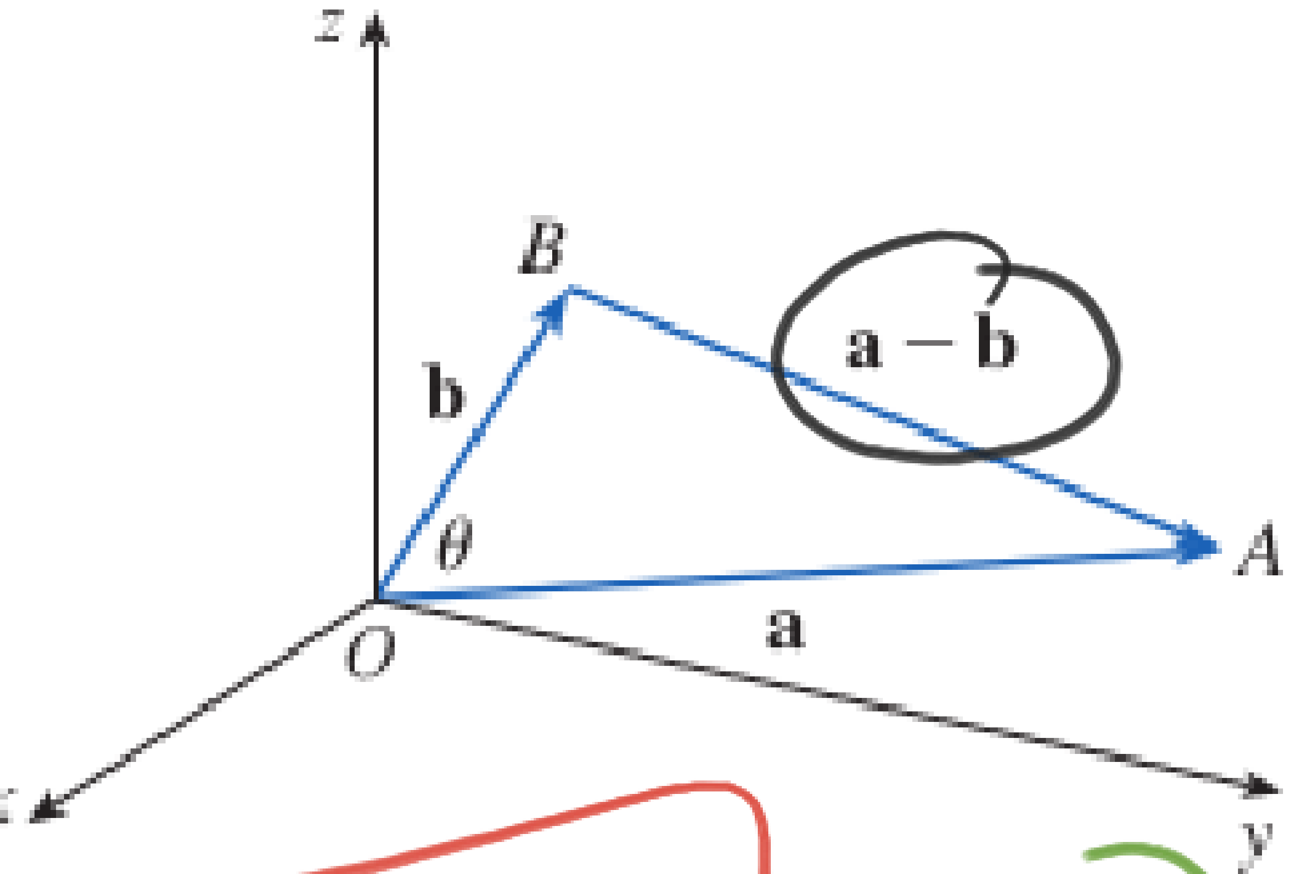
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5. $\mathbf{0} \cdot \mathbf{a} = 0$



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$|AB|^2 = |OA|^2 + |OB|^2 -$$

$$2|OA||OB|\cos\theta$$

$$\boxed{|a-b|^2} = |a|^2 + |b|^2 - 2|a||b|\cos\theta$$

$$a \cdot a = |a|^2 \qquad |a-b|^2 = (a-b)\cdot(a-b)$$

$$= |a|^2 - 2a\cdot b + |b|^2$$

$$-2(a\cdot b) = -2|a||b|\cos\theta$$

$$-2(a \cdot b) = -2|a||b|\cos\theta$$

$$a \cdot b = |a||b|\cos\theta$$

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

3 Theorem

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

6 Corollary

If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

If the vectors **a** and **b** have lengths 4 and 6 , and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

$$a \cdot b = |a||b| \cos \theta$$

$$a \cdot b = (4)(6) \cos \frac{\pi}{3}$$

$$= 24 \left(\tfrac{1}{2}\right)$$

$$= 12$$

Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\cos \theta = \frac{2}{3\sqrt{38}}$$

$$|\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

$$\arccos \frac{2}{3\sqrt{38}} = 83.79145537$$

$\cos^{-1}$

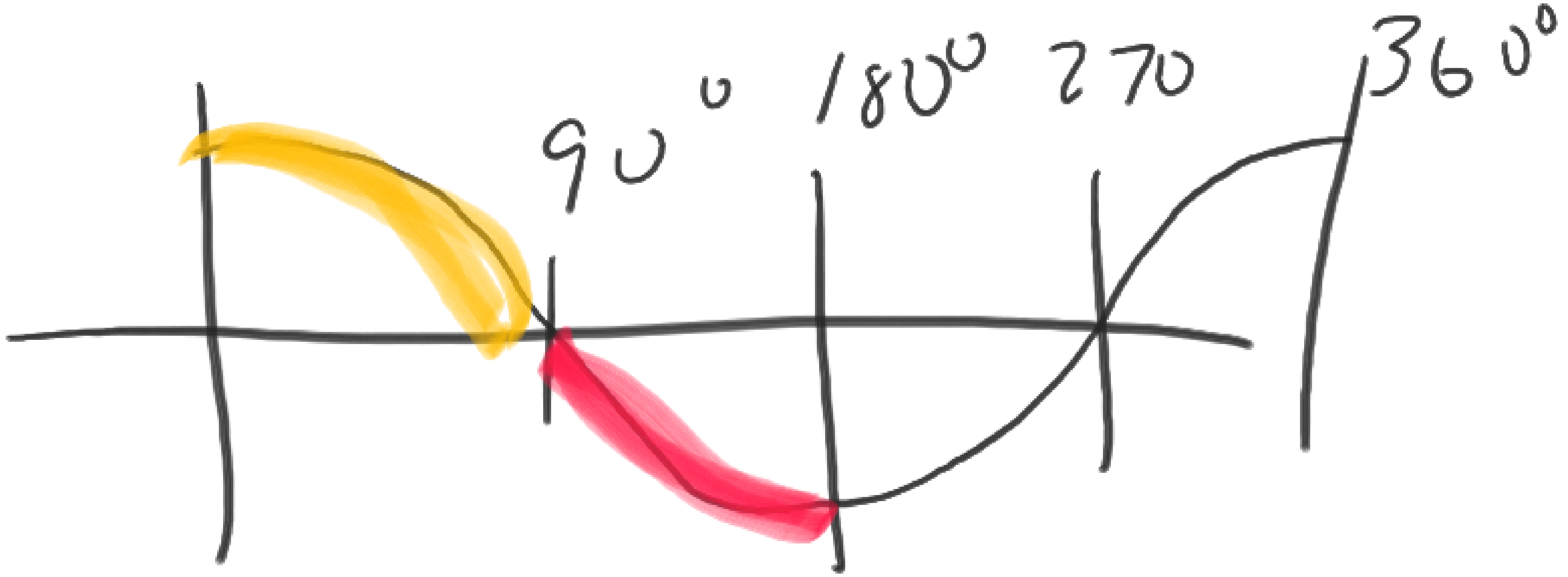
$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

$a \cdot b = 0$ perpendicular

$a \cdot b > 0$ acute $(< 90^\circ)$

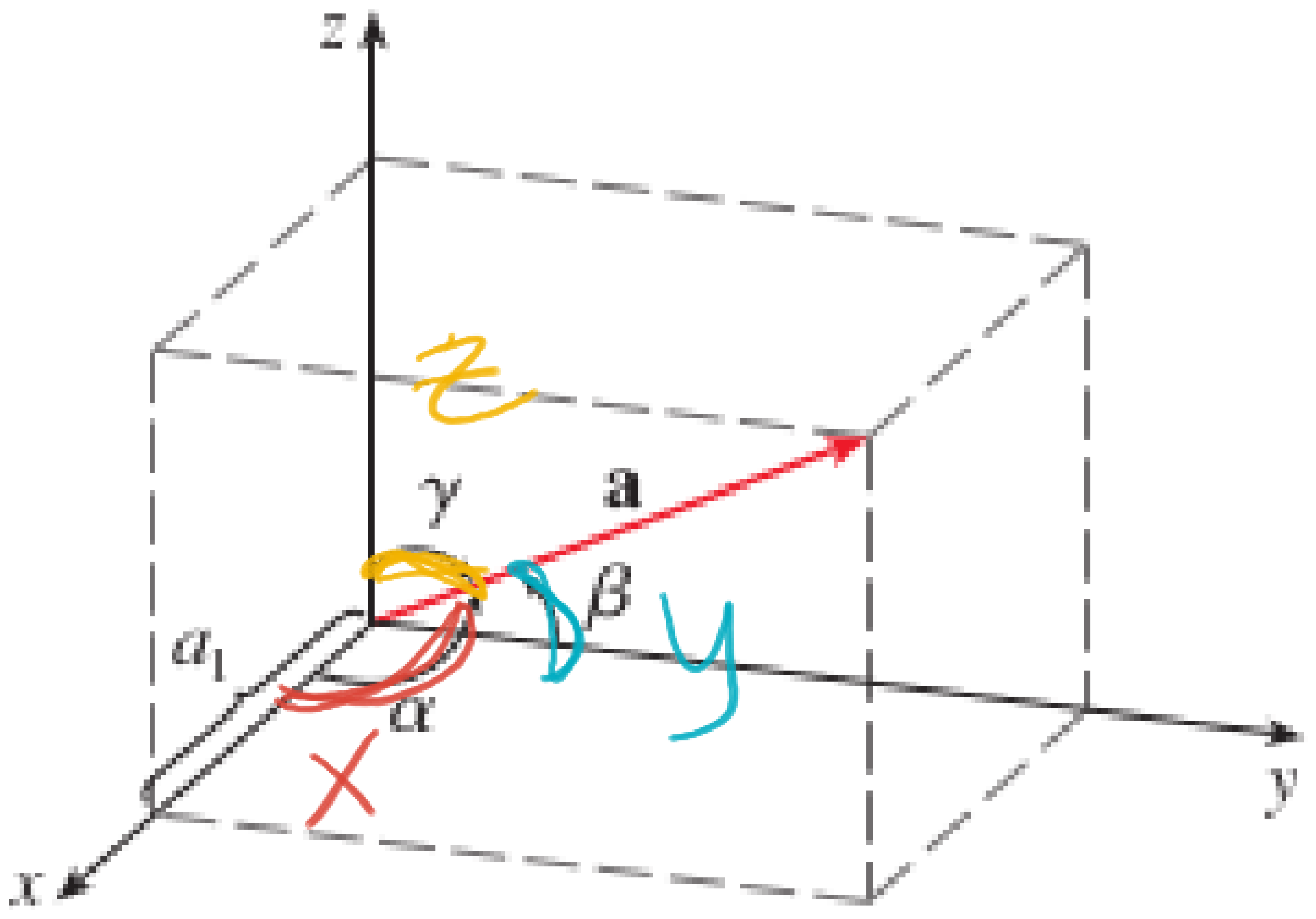
$a \cdot b < 0$ obtuse $(90 \to 180^\circ)$



Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

$$(2)(5) + (2)(-4) + (-1)(2) \quad \text{yes}$$

$$10 \quad - 8 \quad - 2 \quad = 0$$



ls

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

$$\mathbf{a} = \langle\, a_1, a_2, a_3 \,\rangle = \langle |\mathbf{a}| \cos\alpha, |\mathbf{a}| \cos\beta, |\mathbf{a}| \cos\gamma \rangle$$

$$= |\mathbf{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

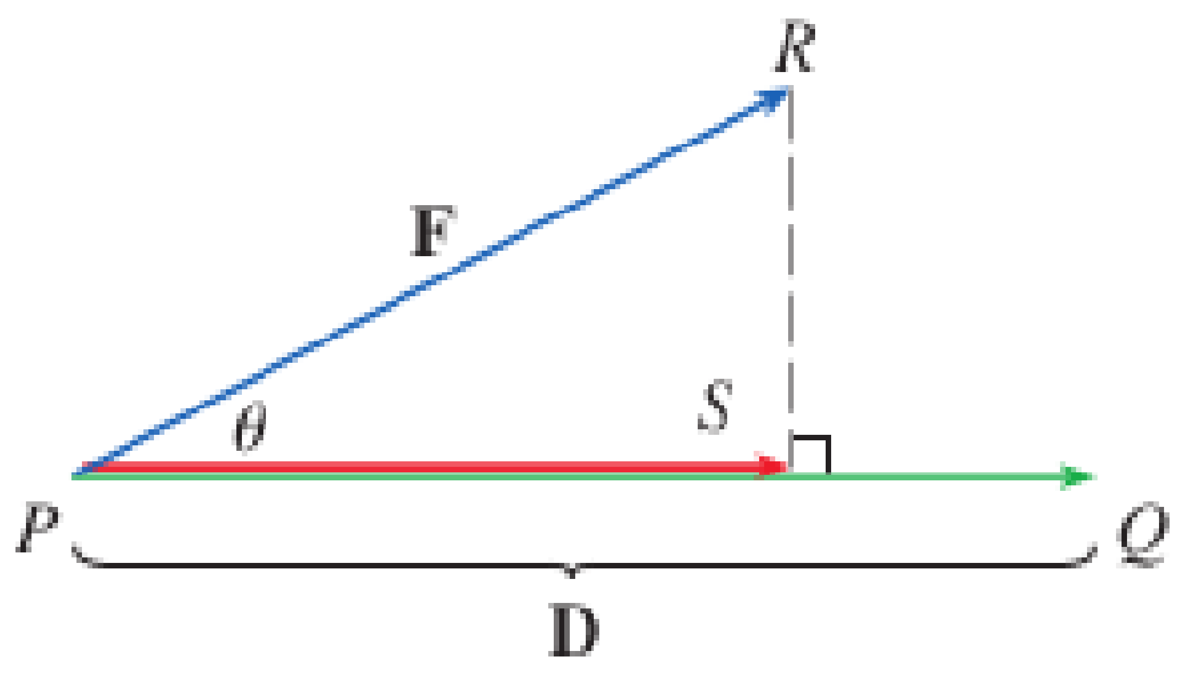
$$\cos x = \frac{1}{\sqrt{14}} = 74.49864043$$

$$|a| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\cos y = \frac{2}{\sqrt{14}} = 57.68846676$$

$$\cos z = \frac{3}{\sqrt{14}} = 36.6992252$$

Figure 6



▸ Details

$$W = F \cdot D$$

But then, from Theorem 3, we have

$$W = |F||D| \cos \theta$$

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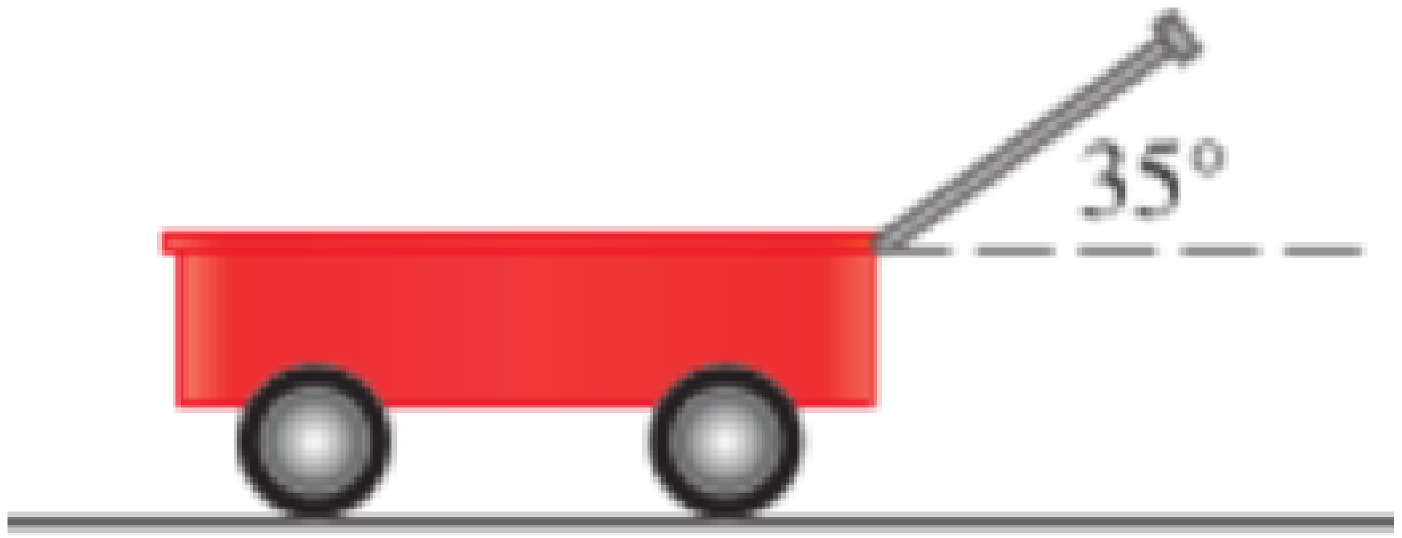
$$W = |\mathbf{F}||\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

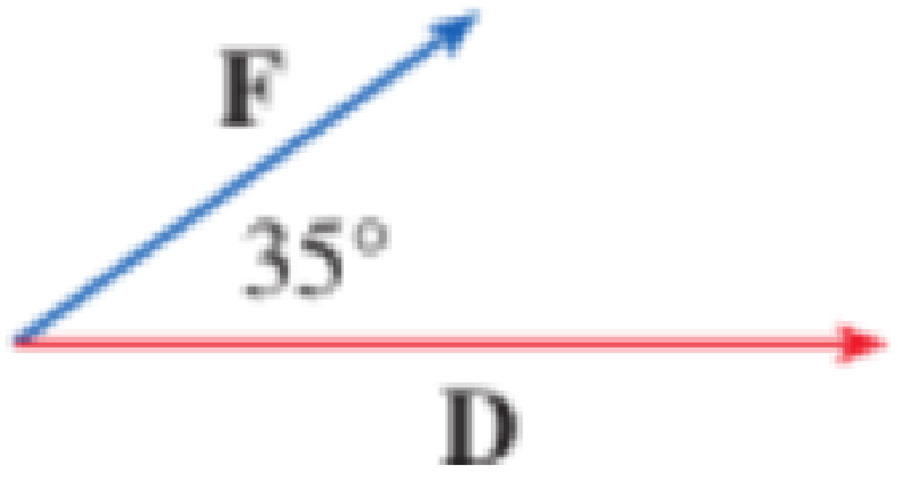
A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.



$W = |F||D| \cos\theta$

$W = (70)(100) \cos 35°$

$W = 5734 \text{ N} \cdot \text{m} = 5734 \text{ J}$



A force is given by a vector $\mathbf{F} = 3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

$\langle 4-2,\ 6-1,\ 2-0 \rangle$

$\langle 2,\ 5,\ 2 \rangle = D$

$$W = \overset{6}{(3)(2)} + \overset{20}{(4)(5)} + \overset{10}{(5)(2)}$$

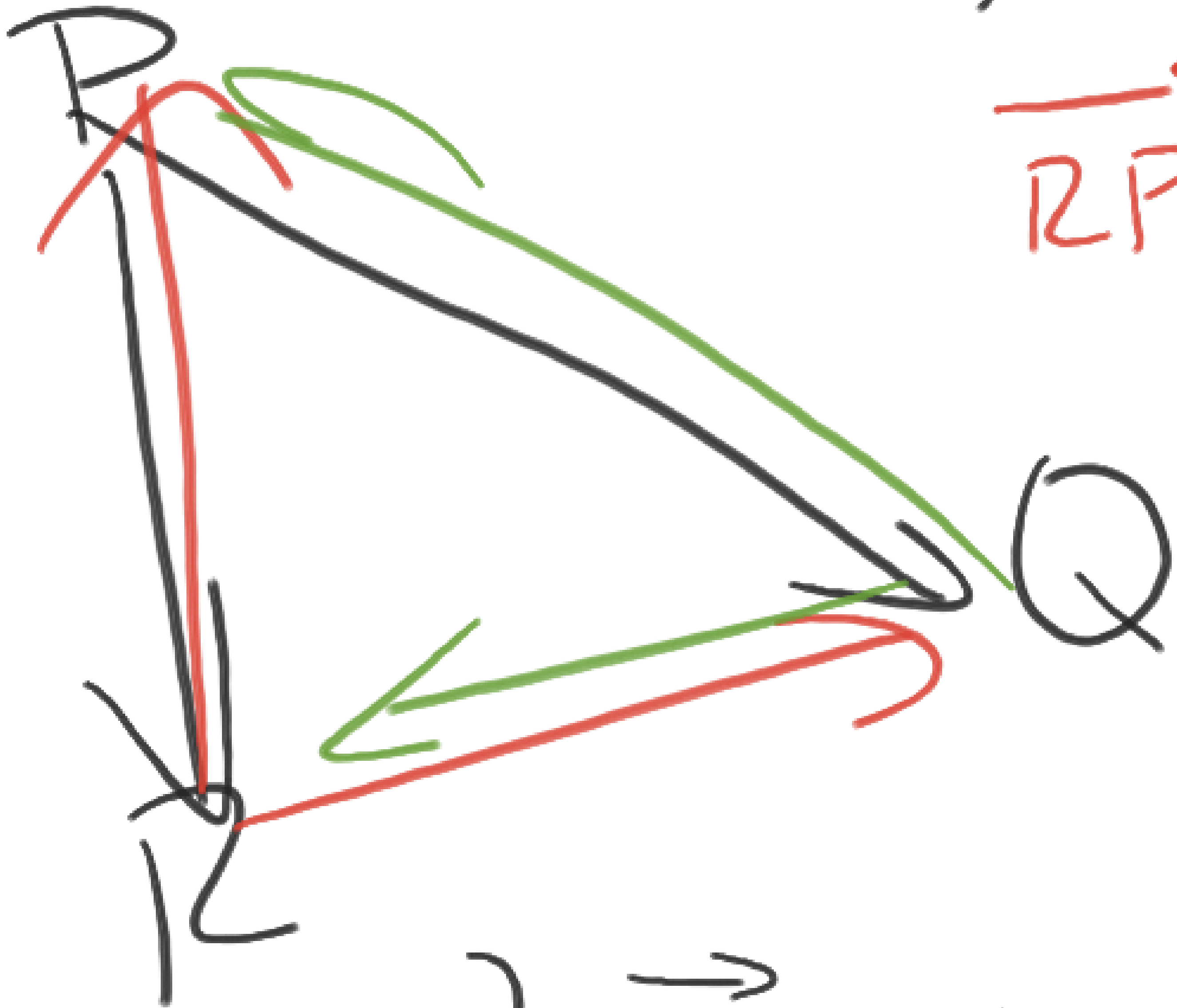
$= 36\,J$

50. A tow truck drags a stalled car along a road. The chain makes an angle of 30 ° with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

$$W = (1500)(1000)\cos 30^\circ$$

$$\approx 1,300,000$$

25. Use vectors to determine whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.



$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 5+3-6 \neq 0$

$\overrightarrow{RP} \cdot \overrightarrow{RQ} = 20-2-6 \neq 0$

$\overrightarrow{QP} \cdot \overrightarrow{QR} = 4+6-2$

$= 0$

$\overrightarrow{PQ} = \langle 1, 3, -2 \rangle$

$\overrightarrow{QP} = \langle -1, -3, 2 \rangle$

$\overrightarrow{PR} = \langle 5, 1, -3 \rangle$

$\overrightarrow{RP} = \langle -5, -1, 3 \rangle$

$\overrightarrow{RQ} = \langle -4, 2, 1 \rangle$

$\overrightarrow{QR} = \langle 4, -2, -1 \rangle$