$, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$(a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j$$

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$ =

$$\boxed{-43i + 13j + k}$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} k$$

$$-15 - 28 \qquad -[(-5) - 8] \qquad 7 - 6$$

$$-43i + 13j + 1k$$

$$\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$a \times b = \begin{vmatrix} i & j & k \\ 0 & 2 & -4 \\ -1 & 3 & 1 \end{vmatrix} = 14i + 4j + 2k$$

$$= \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 0 & -4 \\ -1 & 1 \end{vmatrix} j + \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} k$$

$$2-(-12) \quad - \quad 0-4 \qquad\qquad 0-(-2)$$

$$14i \quad + 4j \quad + 2k$$

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$

$$= 0$$

9 Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

10 Corollary

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

$$\overrightarrow{PQ} = a = -3i + j - 7k$$

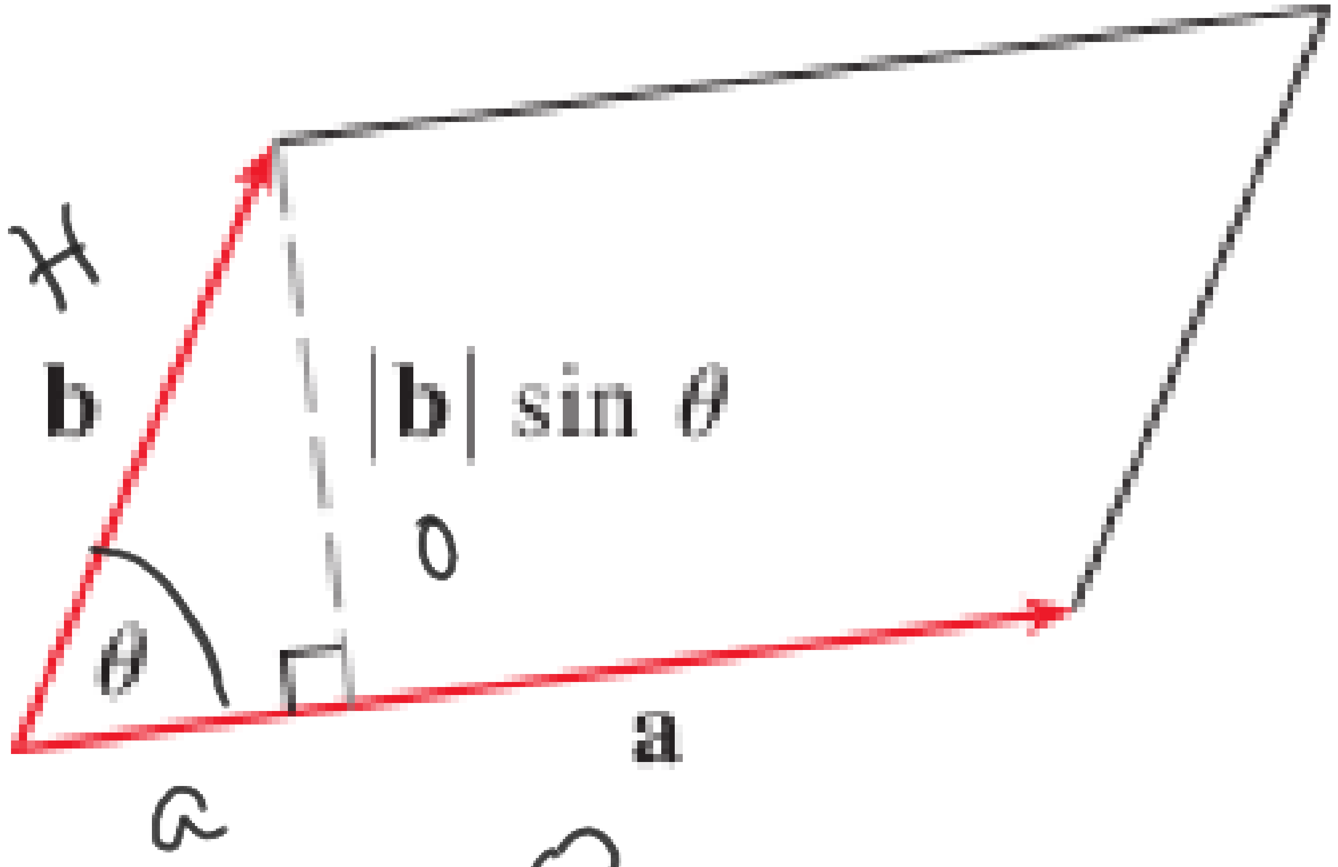
$$\overrightarrow{PR} = b = 0i - 5j - 5k$$

$$\begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} i - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} j + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix} k$$

$$-5 - 35 \quad - (15 - 0) \quad + (15 - 0)$$

$$= -40i - 15j + 15k$$



$$|\mathbf{a} \times \mathbf{b}| = \boxed{|\mathbf{a}||\mathbf{b}| \sin \theta}$$

$$\sin \theta = \frac{O}{b}$$

$$|b| \sin \theta = O \to h$$

$$A = bh$$

$$= |a||b| \sin \theta$$

$$A = \frac{|a||b| \sin \theta}{2}$$

Trangle

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

$-40\,\hat{i} - 15\,\hat{j} + 15\,\hat{k}$

$$|a \times b| = \sqrt{(-40)^2 + (-15)^2 + (15)^2} = 5\sqrt{82}$$

$$\approx 45.277/2$$

$$\frac{5}{2}\sqrt{82} \approx \boxed{22.638}$$

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

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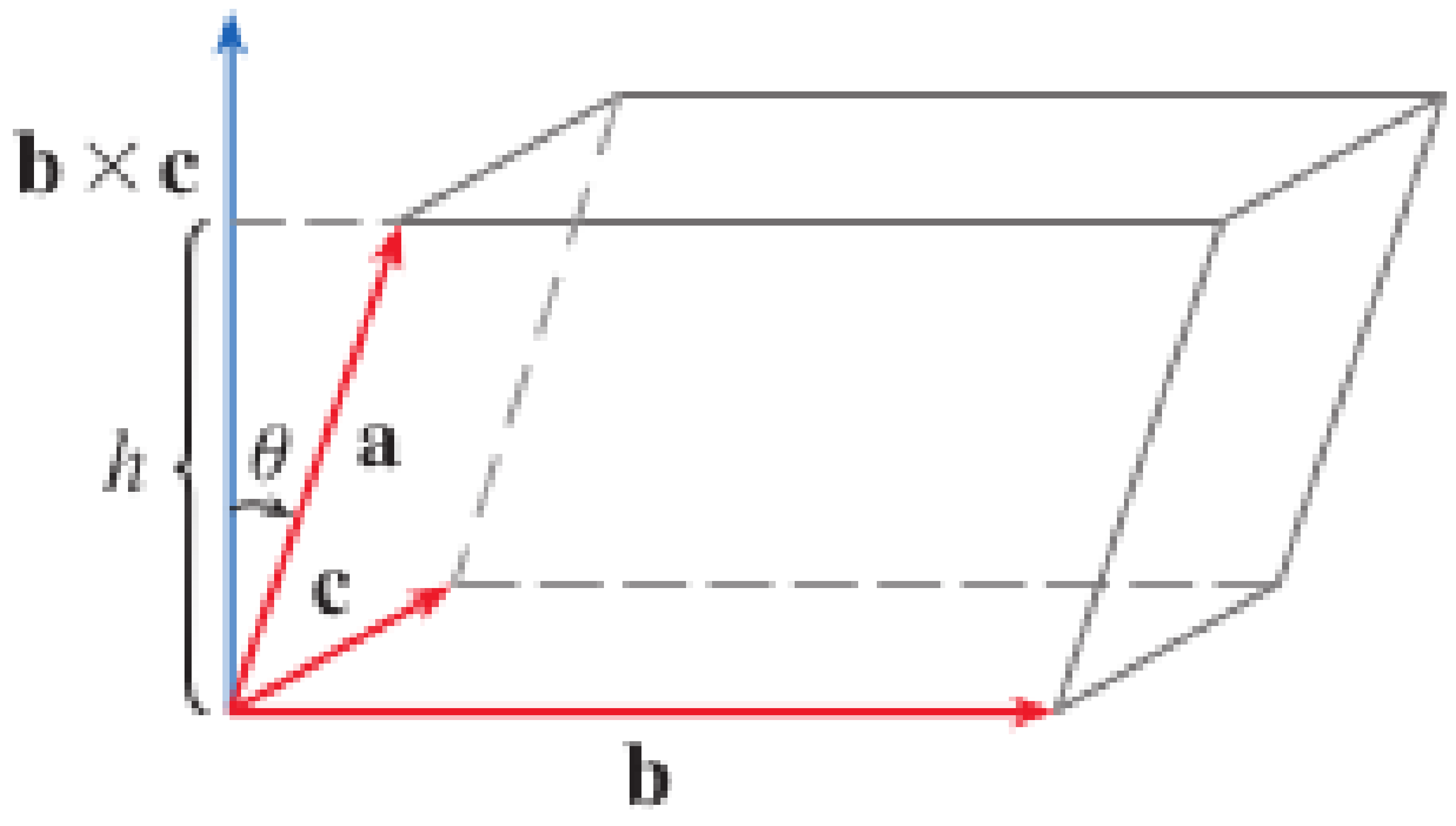
$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\
&= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\
&= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
\end{aligned}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$



$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\,\theta| =$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\,\theta$$

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

parallelepiped

Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

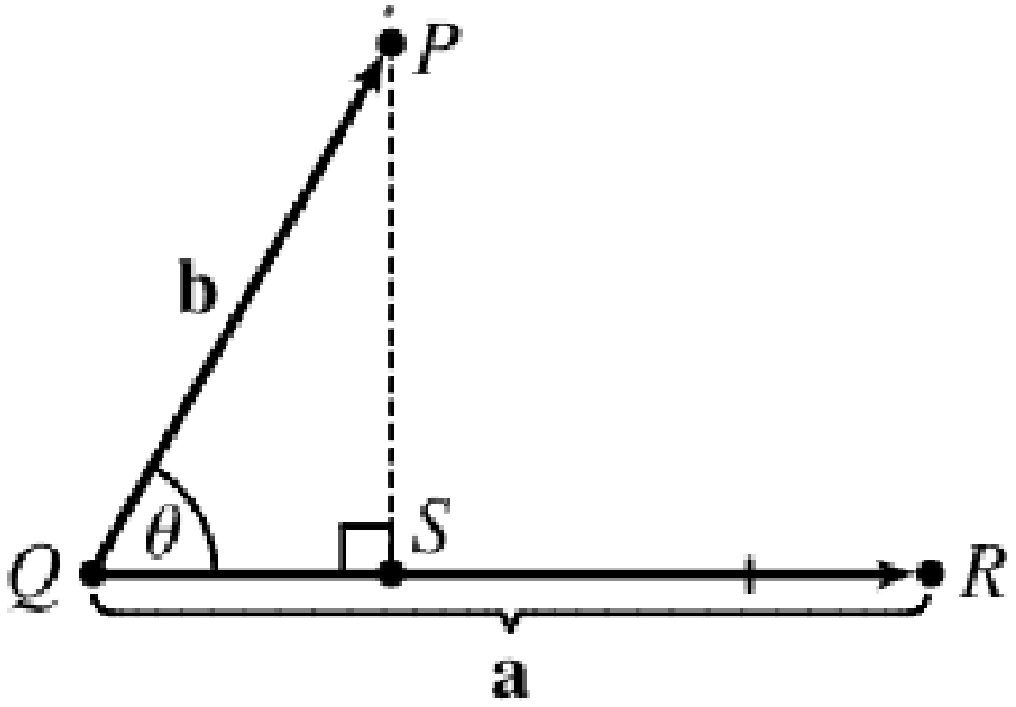
$$a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 0$$

$$1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

DISTANCE FROM A POINT TO A LINE

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

b. Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.



$\overrightarrow{QR} = a = \langle -1, -2, -1 \rangle$

$\overrightarrow{QP} = b = \langle 1, -5, -7 \rangle$

$a \times b \Rightarrow \langle 9, -8, 7 \rangle.$

$d = \dfrac{\sqrt{194}}{\sqrt{6}} = \boxed{\sqrt{\dfrac{97}{3}}}$

$$\overrightarrow{QR} = a = \langle -1, -2, -1 \rangle$$

$$\overrightarrow{QP} = b = \langle 1, -5, -7 \rangle$$

$$\begin{vmatrix} i & j & k \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} -2 & -1 \\ -5 & -7 \end{vmatrix} i - \begin{vmatrix} -1 & -1 \\ 1 & -7 \end{vmatrix} j + \begin{vmatrix} -1 & -2 \\ 1 & -5 \end{vmatrix} k$$

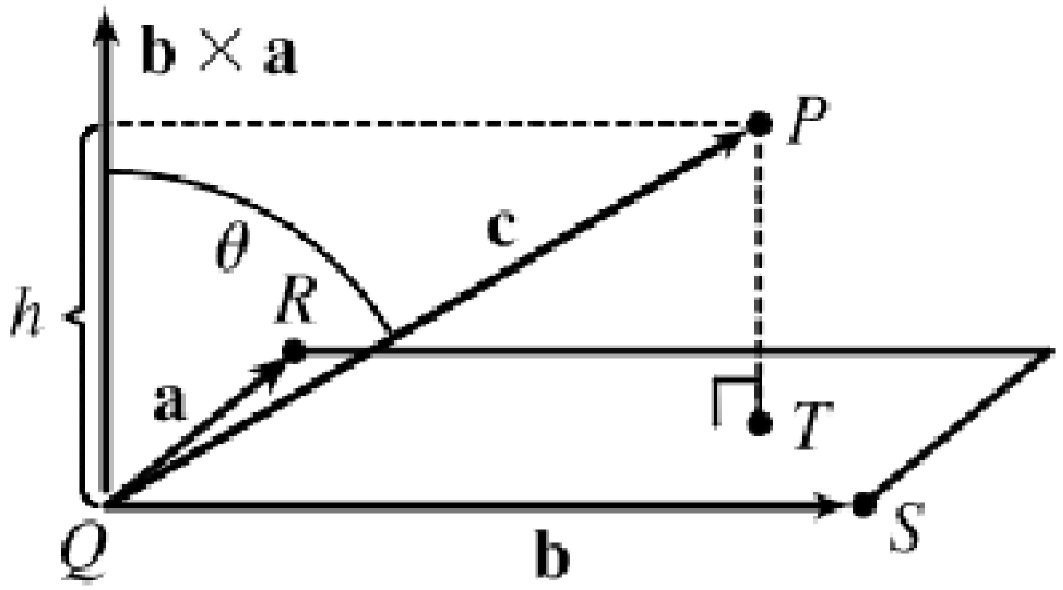
$$14 - 5 \qquad 7 - (-1) \qquad 5 - (-2)$$

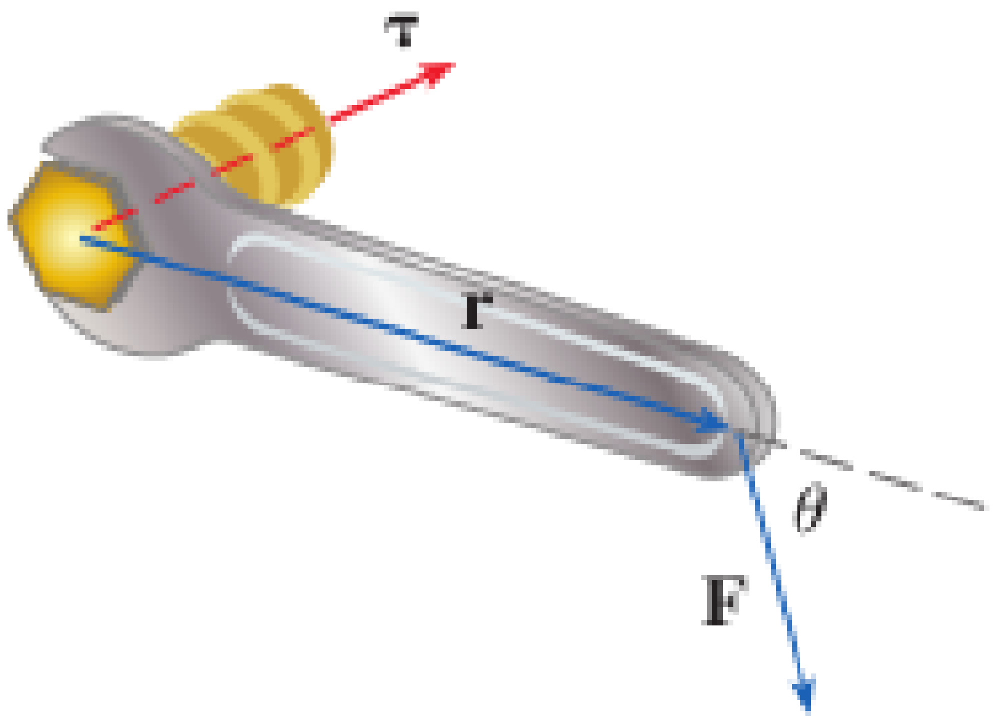
$$9i - 8j + 7k$$

DISTANCE FROM A POINT TO A PLANE

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.



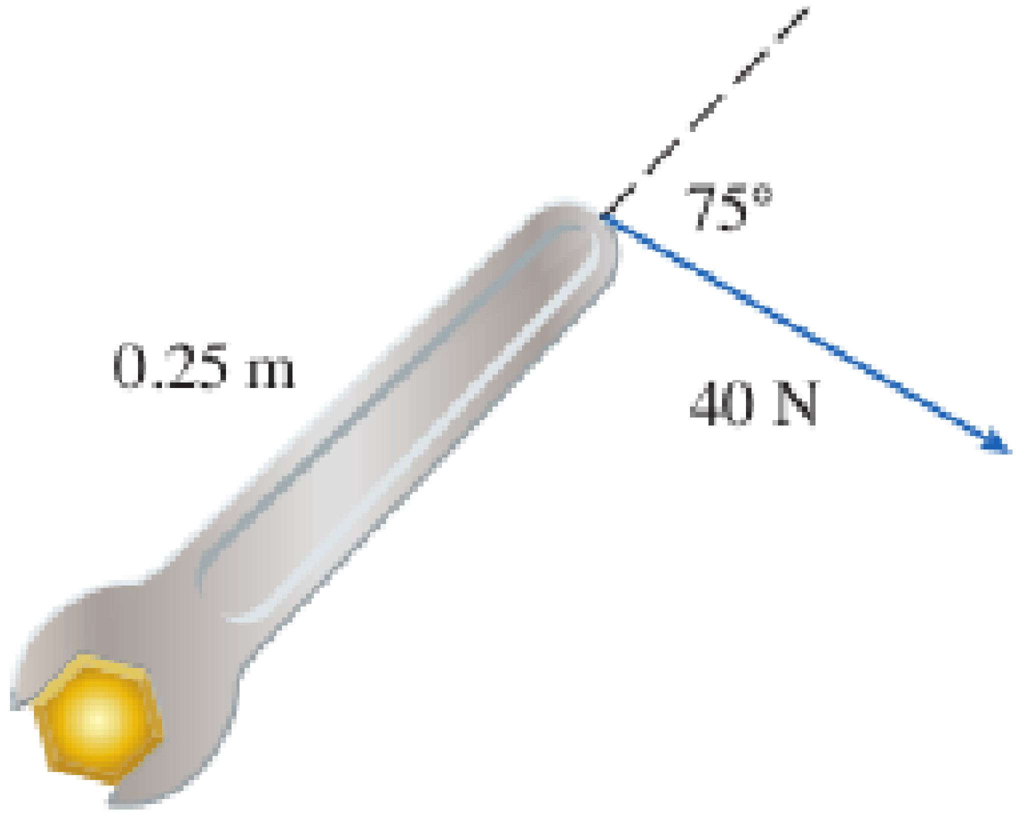


$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown in Figure 5.

Find the magnitude of the torque about the center of the bolt.



$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

$$|\tau| = (0.25 m)(40 N) \sin 75^\circ$$

$$= 10 \sin 75^\circ \approx 9.66 \text{ N} \cdot \text{m}$$

41. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 N · m of torque to the bolt.

$a \langle 0, 3, -4 \rangle \qquad b \langle 0, 0.3, 0 \rangle$

$\cos \theta = \dfrac{a \cdot b}{|a||b|} = \dfrac{0.9}{(5)(0.3)} \to \theta = 53.13$

$100 = 0.3 \boxed{|F|} \sin \theta$

$100 = 0.3 (\sin 53.13) |F|$

$$|\mathbf{F}| \approx \frac{100}{0.3 \sin 53.1^\circ} \approx 417 \text{ N.}$$