7.2 Trigonometric Integrals

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\csc^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Find $\int \sin^5 x \cos^2 x \, dx$.

$u = \cos x$

$du = -\sin x \, dx$

$$\int \sin x \, \boxed{\boxed{\sin^4(x)} \cos^2(x)} \, dx$$

$(1-\cos^2 x)^2$

$(1-u^2)^2(u^2)$

$(1-2u^2+u^4)$

$$-\int (1-u^2)^2(u^2)\,du = -\int u^2 - 2u^4 + u^6 \, du$$

$$-\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C = -\left(\frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7}\right.$$

$$\int \sin^{2k+1}x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. See Example 2.

[Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\boxed{\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)} \qquad \boxed{\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)}$$

See Examples 3 and 4.

It is sometimes helpful to use the identity

$$\boxed{\sin x \cos x = \tfrac{1}{2} \sin 2x}$$

$$\cos^2(a) = 1 + \cos 2a$$

$a = 2x$

$$\cos^2(2x) = 1 + \cos 2(2x)$$

$$= 1 + \cos 4x$$

Find $\int \sin^4 x\, dx.$ $= \frac{1}{2}\int (1-\cos 2x)^2\, dx$

$(\sin^2 x)^2$

$\frac{1}{2}\int 1 - 2\cos 2x + \cos^2 2x\ dx$

$\frac{1}{2}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\ dx$

$1/2 + 1/2 \cos 4x$

$\frac{1}{2}\int \frac{3}{2} - 2\cos 2x + 1/2 \cos 4x\ dx$

$$\frac{1}{2}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \, dx$$

$$\frac{1}{2}\left[\frac{3}{2}x - 2\left(\frac{1}{2}\right)\sin 2x + \frac{1}{2}\left(\frac{1}{4}\right)\sin 4x\right] + C$$

$$\frac{3}{4}x - \frac{1}{2}\sin 2x + \frac{1}{16}\sin 4x + C$$

Evaluate $\int \tan^6 x \sec^4 x \, dx.$

$\tan^2 x + 1 = \sec^2 x$

$\int \tan^6 x \, \boxed{\sec^2 x} \, \boxed{\sec^2 x \, dx}$

$(\tan^2 x + 1)$

$u = \tan x$

$\boxed{du =} \sec^2 x \, dx$

$\int u^6 (u^2 + 1) \, du = \int u^8 + u^6 \, du$

$$\frac{u^9}{9} + \frac{u^7}{7} + C = \boxed{\frac{\tan^9(x)}{9} + \frac{\tan^7(x)}{7} + C}$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$. See Example 5.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$u = \sec x$

$du = \tan x \sec x$

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$. See Example 6.

$$\int \tan x \, dx = \ln |\sec x| + C \qquad \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \tan^3 x\,dx. = \int \tan x \underset{(\sec^2 x - 1)}{(\tan^2 x)}\,dx$$

$$\int \tan x \sec^2 x\,dx - \boxed{\int \tan x \;\; dx}$$

$$\int \sec x\,(\tan x \sec x)\,dx$$

$u = \sec x$

$du = \tan x \sec x\,dx$

$$\int u\,du = \frac{u^2}{2}$$

$$= \boxed{\frac{\sec^2 x}{2} - \ln|\sec x| + C}$$

Find $\int \sec^3 x \, dx.$ $= \int \sec x \sec^2 x \, dx$

$u = \sec x \qquad v = \tan x$

$du = \sec x \tan x \, dx \qquad dv = \sec^2 x \, dx$

$\sec x \tan x - \int \tan x \sec x \tan x \, dx$

$\sec x \tan x - \int \tan^2 x \sec x \, dx$

$$\sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$(\sec^2 x - 1)$$

$$-\left[\int \sec^3 x \, dx - \int \sec x \, dx\right]$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2}$$

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Evaluate $\int \sin 4x \cos 5x \, dx$.

$$= \frac{1}{2}\int \sin(-x) + \sin(9x) \, dx$$

$$= \frac{1}{2}\left[(-1)(-\cos(-x)) + \frac{1}{9}(-\cos 9x)\right] + C$$

$$= \frac{1}{2}\left[(-1)(-\cos(-x)) + \frac{1}{9}(-\cos 9x)\right] + C$$

$$= \frac{1}{2}\cos(-x) - \frac{1}{18}\cos(9x) + C$$

$$\int \cos^3(t/2)\sin^2(t/2)\,dt$$

$$m = t/2$$
$$dm = 1/2\, dt$$
$$2\,dm = dt$$

$$2\int \cos^3(m)\sin^2(m)\,dm$$

$$2\int \cos^2(m)\sin^2(m)\cos(m)\,dm$$

$$2\int (1-\sin^2(m))\sin^2(m)\,\boxed{\cos(m)\,dm}$$

$$u = \sin(m) \qquad \boxed{du} = \cos(m)\,dm$$

(2) $\int (1-u^2)\, u^2 \, du$

$$2\int u^2 - u^4 \, du$$

$$2\left[\frac{u^3}{3} - \frac{u^5}{5}\right] = C + \frac{2\sin^3(t/2)}{3} - \frac{2\sin^5(t/2)}{5}$$

$$u = \sin\left(t/2\right)$$

$$du = \frac{1}{2}\cos\left(t/2\right)\,dt$$

$$\boxed{2\,du} = \cos\left(t/2\right)\,dt$$

$$\int \tan^2 x \cos^3 x \, dx$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$$

$$\int \frac{\sin^2 x}{\cos^2 x} \cos^3 x \, dx$$

$$u = \sin x$$

$$\int \sin^2 x \cos x \, dx$$

$$du = \cos x \, dx$$

$$\int u^2 \, du = \frac{u^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}$$

$$\int \tan^2 x \quad \sin^3 x$$

$$\int \frac{\sin^2 x}{\cancel{\cos^2 x}} \sin^3 x = \frac{\sin^5 x}{\cos^2 x}$$

$$\tan^2 x \sin x \left(1 - \cancel{\cos^2 x}\right)$$