7.4 Integration of Rational Functions by Partial Fractions

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - 1(x-1)}{(x-1)(x+2)}$$

$$= (2x+4-x+1)/(x-1)(x+2)$$

$$= \boxed{\frac{x+5}{x^2+x-2}}$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

$$\int \frac{2}{x-1}$$

$u = x-1$

$du = dx$

$$\int \frac{2}{u} = 2\ln|u|$$

$2\ln|x-1|$

1 $\int \frac{x^3 + x}{x - 1}\, dx.$

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 + 0x^2 + x + 0 \\
 & -x^3 + x^2 \\
\hline
 & x^2 + x + 0 \\
 & -x^2 + x \\
\hline
 & 2x + 0 \\
 & -2x + 2 \\
\hline
 & 2
\end{array}
$$

$$\int x^2 + x + 2 + \frac{2}{x-1}\, dx$$

$$\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

CASE I The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

$2x^3+3x^2-2x = x(2x^2+3x-2)$

$x(x+2)(2x-1)$

$2x^2+4x \mid -x-2$

$2x(x+2) \quad -1(x+2)$

$(x+2)(2x-1)$

Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx.$

$$\int \frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{\overset{1/2}{A}}{x} + \frac{\overset{-1/10}{B}}{x+2} + \frac{\overset{1/5}{C}}{2x-1}$$

$$x^2+2x-1 = A(x+2)(2x-1) + B(x)(2x-1) + C(x)(x+2)$$

$x=0$ $\quad -1 = A(2)(-1) \to -1 = -2A \to A = 1/2$

$x=-2$ $\quad -1 = B(-2)(-5) \to -1 = 10B \to B = -1/10$

$x=1/2$ $\quad 1/4 = C(1/2)(5/2) \to 1/4 = \frac{5}{4}C \to C = 1/5$

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

$$\int \frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{10}\ln|2x-1| + C$$

$$\int \frac{1/2}{x} - \frac{1/10}{x+2} + \frac{1/5}{2x-1}$$

CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.**
Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

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$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

but we prefer to work out in detail a simpler example.

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx. = \int x+1 + \frac{4x}{x^3-x^2-x+1}\,dx$

$$\begin{array}{r}
x + 1 \\
x^3 - x^2 - x + 1 \overline{\smash{)}\, x^4 + 0x^3 - 2x^2 + 4x + 1} \\
-x^4 + x^3 + x^2 \mp x \\
\hline
x^3 - x^2 + 3x + 1 \\
-x^3 + x^2 + x \mp 1 \\
\hline
4x
\end{array}$$

$x^3 - x^2 \mid -x + 1$

$x^2(x-1) - 1(x-1)$

$(x^2-1)(x-1)$

$(x-1)(x+1)(x-1)$

$$\int x+1+\boxed{\frac{4x}{(x-1)^2(x+1)}}\,dx$$

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$x=-1$ $\quad -4=C(4) \rightarrow C=-1$

$x=1$ $\quad 4=B(2) \rightarrow B=2$

$x=0$ $\quad 0=A(-1)+B+C \rightarrow 0=-A+1$

$-1=-A \rightarrow A=1$

$$\int x+1+\boxed{\frac{4x}{(x-1)^2(x+1)}}\,dx$$

$$\int x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\,dx$$

$$\boxed{\frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+C}$$

CASE III $Q(x)$ **contains irreducible quadratic factors, none of which is repeated.** If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \tag{9}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x - 2)(x^2 + 1)(x^2 + 4)} = \boxed{\frac{A}{x - 2}} + \frac{Bx + C}{x^2 + 1} + \boxed{\frac{Dx + E}{x^2 + 4}}$$

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx. = \int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx$$

$$\frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$A = 1$ $C = -1$

$B = 1$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)(x)$$

$$2x^2 - x + 4 = Ax^2 + A(4) + Bx^2 + Cx$$

$$2x^2 - x + 4 = (A+B)(x^2) + (C)(x) + A(4)$$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx. = \int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx$$

$$\int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx = \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$$\boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

$$\int \frac{x}{x^2+4}$$

$$u = x^2+4$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\int \frac{1/2}{u}$$

$$\frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+4|$$

CASE IV $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (9), the sum

$$\boxed{11} \qquad \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (11) can be integrated by using a substitution or by first completing the square if necessary.

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$\begin{aligned}
-x^3 + 2x^2 - x + 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\
&= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\
&= \overset{0}{(A + B)}x^4 + \overset{-1}{C}x^3 + \overset{2}{(2A + B + D)}x^2 + \overset{-1}{(C + E)}x + \overset{1}{A}
\end{aligned}$$

$A = 1$

$B = -1$

$C = -1$

$D = 1$

$E = 0$

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2}$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C$$

$$\int \frac{x}{(x^2+1)^2}$$

$u = x^2+1$

$du = 2x\,dx$ $\quad \frac{1}{2}\int \frac{1}{u^2}$

$\frac{1}{2}\,\frac{-1}{u}$

$$\int \frac{5x}{x(x^4+8)^2} \Rightarrow \frac{Ax^3+Bx^2+Cx+D}{x^4+8}$$

$$= \frac{A}{x} + \frac{Bx^3+Cx^2+Dx+E}{x^4+8}$$

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$

$$= \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + x +}$$

It would be extremely tedious to work out by hand the numerical values of the coefficients in Example 7. Most computer algebra systems, however, can find the numerical values very quickly:

$$A = -1, \quad B = \tfrac{1}{8}, \quad C = D = -1,$$

$$E = \tfrac{15}{8}, \quad F = -\tfrac{1}{8}, \quad G = H = \tfrac{3}{4},$$

$$I = -\tfrac{1}{2}, \quad J = \tfrac{1}{2}$$