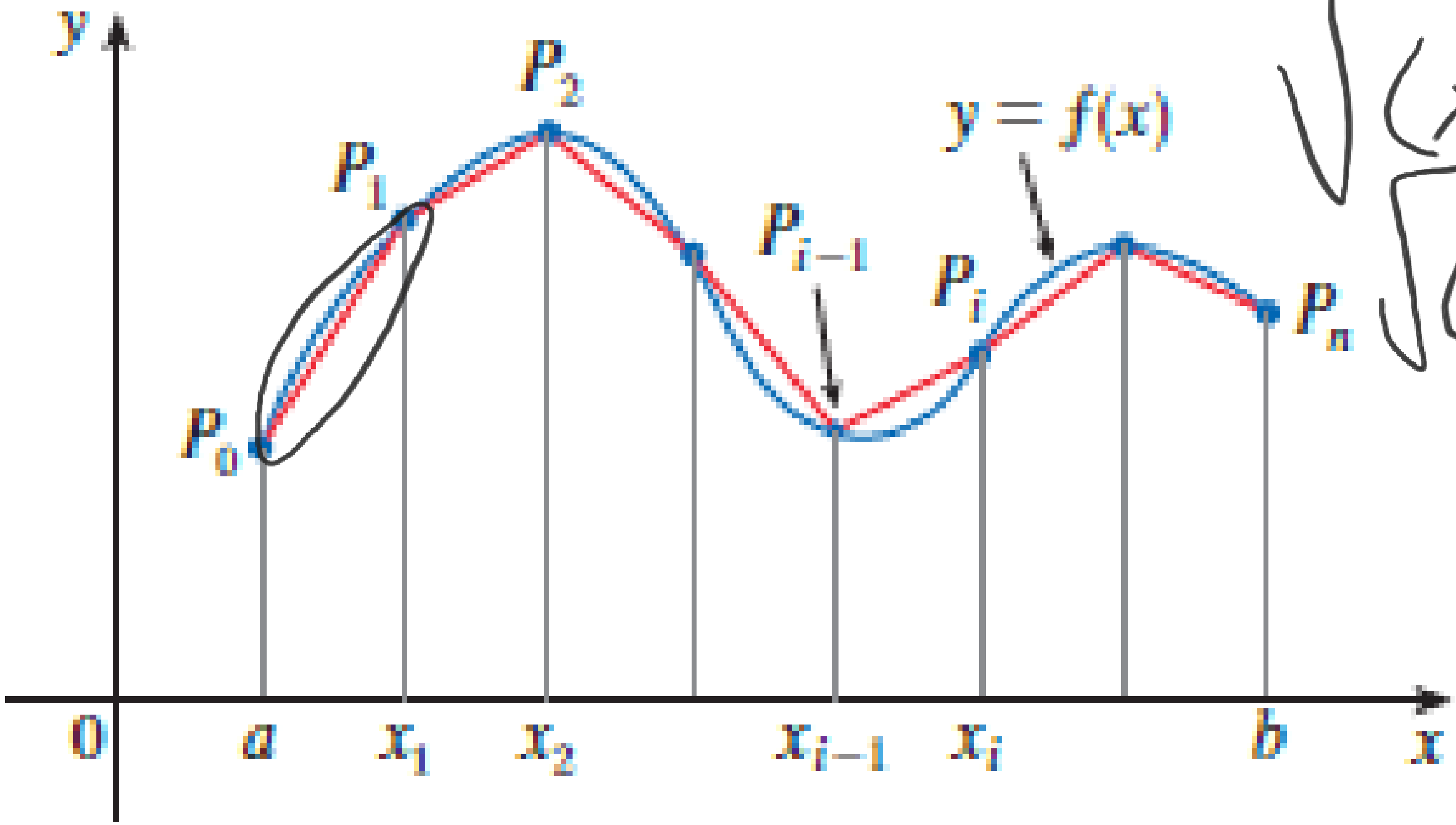
8.1 Arc Length

$$\sqrt{x^2+y^2}$$

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\sqrt{(\Delta x)^2+(\Delta y)^2}$$



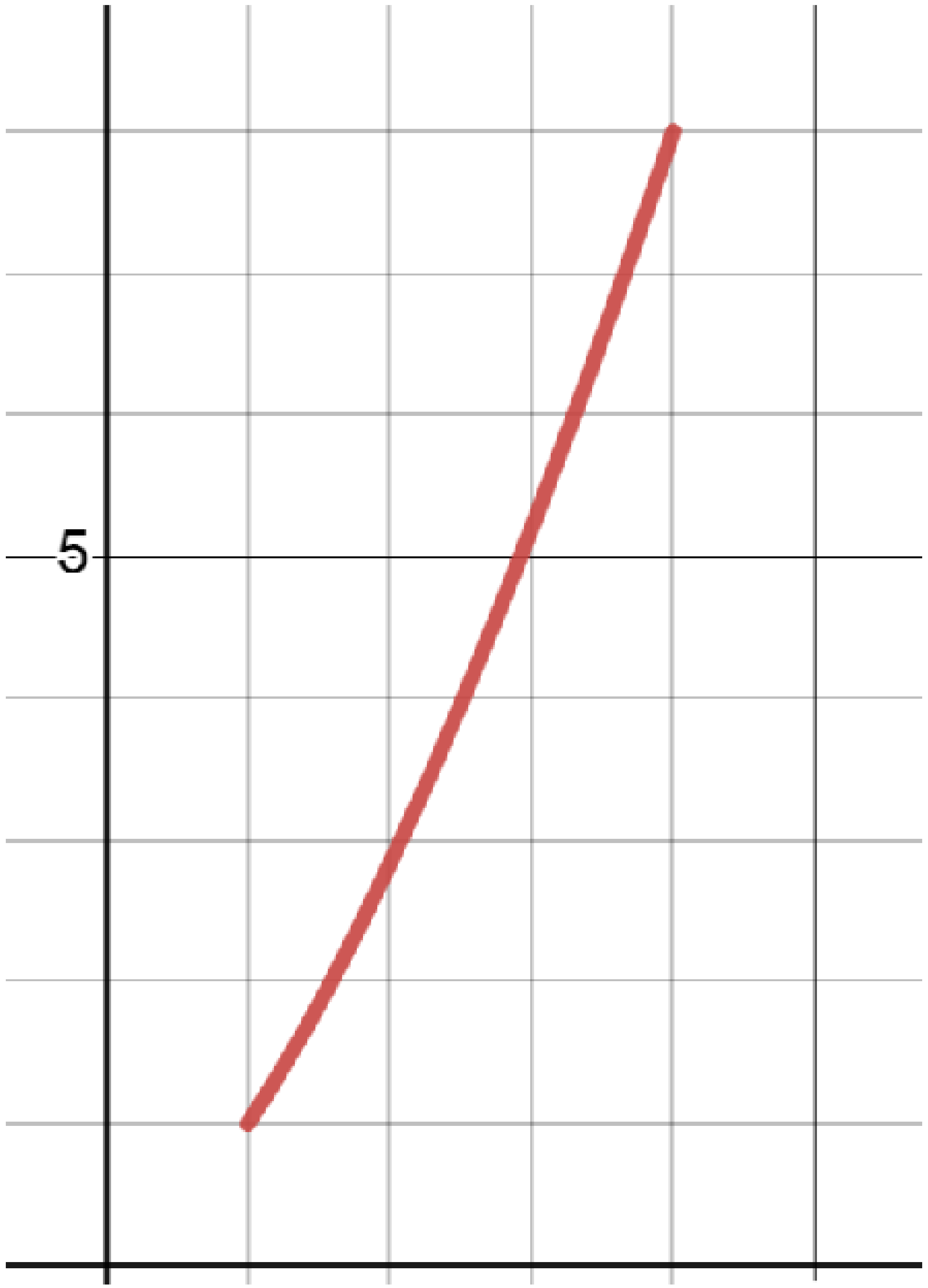
2 **The Arc Length Formula** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

3
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (See Figure 5.)



$y^2 = x^3 \rightarrow y = x^{3/2}$

$L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2} \rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$

$\int_1^4 \sqrt{1 + 9/4x}\, dx$

$$\int_1^4 \sqrt{1+\tfrac{9}{4}x}\;dx$$

$u = 1 + \tfrac{9}{4}x$

$du = \tfrac{9}{4}\,dx$

$\tfrac{4}{9}\,du = dx$

$$\frac{4}{9}\int_1^4 u^{1/2}\,du$$

$$\frac{4}{9}\left(\frac{2}{3}u^{3/2}\right) \rightarrow \frac{8}{27}\left[\left(1+\tfrac{9}{4}x\right)^{3/2}\right]_1^4$$

$$\boxed{\frac{8}{27}\left(10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right)}$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

$x = y^2 \rightarrow dx/dy = 2y \rightarrow \left(\frac{dx}{dy}\right)^2 = 4y^2$

$\int_0^1 \sqrt{1 + 4y^2}\, dy$

$(1)^2 + (2y)^2$

$y = \frac{1}{2}\tan\theta$

$dy = \frac{1}{2}\sec^2\theta\, d\theta$

$y = 0 \rightarrow \theta = 0$

$y = 1 \rightarrow 2 = \tan\theta$

$\tan^{-1} 2 = \theta$

$$\int_0^{\tan^{-1}2} \sqrt{1+(\cancel{2(\tfrac{1}{2}}\tan\theta))^2}\left(\frac{1}{2}\sec^2\theta\right)d\theta$$

$$\sqrt{\cancel{1+\tan^2\theta}\ \sec^2\theta}$$

$$\sec\theta$$

$$\frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta \, d\theta = \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\ \tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}2}$$

$$\frac{1}{4}\left[\sec a\ \tan a + \ln|\sec a + \tan a|\right]$$

$$\frac{1}{4}\left[\sec a \tan a + \ln\left|\sec a + \tan a\right|\right] \qquad a = \tan^{-1} 2$$

$$\tan a = 2$$

$$\sec^2 a = 1 + \tan^2 a$$

$$\sec^2 a = 5 \rightarrow \sec a = \sqrt{5}$$

$$\boxed{\frac{1}{4}\left(2\sqrt{5} + \ln\left|2 + \sqrt{5}\right|\right)}$$

EXAMPLE 3

(a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $\left(2, \frac{1}{2}\right)$.
(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

$$xy = 1 \rightarrow y = \frac{1}{x} \rightarrow \frac{dy}{dx} = \frac{-1}{x^2} \rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^4}$$

$$\int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$

$$= 1.13209039331$$

11. $y = \frac{2}{3}(1 + x^2)^{3/2}, \quad 0 \leqslant x \leqslant 1$

$$\frac{dy}{dx} = \cancel{\frac{2}{3}}\left(\cancel{\frac{3}{2}}(1+x^2)^{1/2}\right)(2x)$$

$$= 2x(1+x^2)^{1/2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + 4x^2(1+x^2)$$

$$= 1+4x^2+4x^4$$

$$\int_0^1 \sqrt{4x^4+4x^2+1}\, dx$$

$$\int_0^1 \sqrt{(2x^2+1)^2}\, dx$$

$$\int_0^1 2x^2+1 \, dx$$

$$\int_0^1 2x^2+1 \, dx$$

$$\left.\frac{2x^3}{3}+x\right]_0^1 = \frac{2}{3}+1 = \boxed{\frac{5}{3}}$$

14. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2}, \quad 1 \leqslant y \leqslant 2$

$$X = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$$

$$\frac{dx}{dy} = \frac{4}{8}y^3 + \frac{-2}{4}y^{-3}$$

$$= \frac{1}{2}y^3 - \frac{1}{2}y^{-3}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{2}y^3 - \frac{1}{2}y^{-3}\right)^2$$

$$= 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6}$$

$$\boxed{= \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4}y^{-6}}$$

$$= \sqrt{\left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)^2}$$

$$\int_1^2 \frac{1}{2}y^3 + \frac{1}{2}y^{-3} \, dy$$

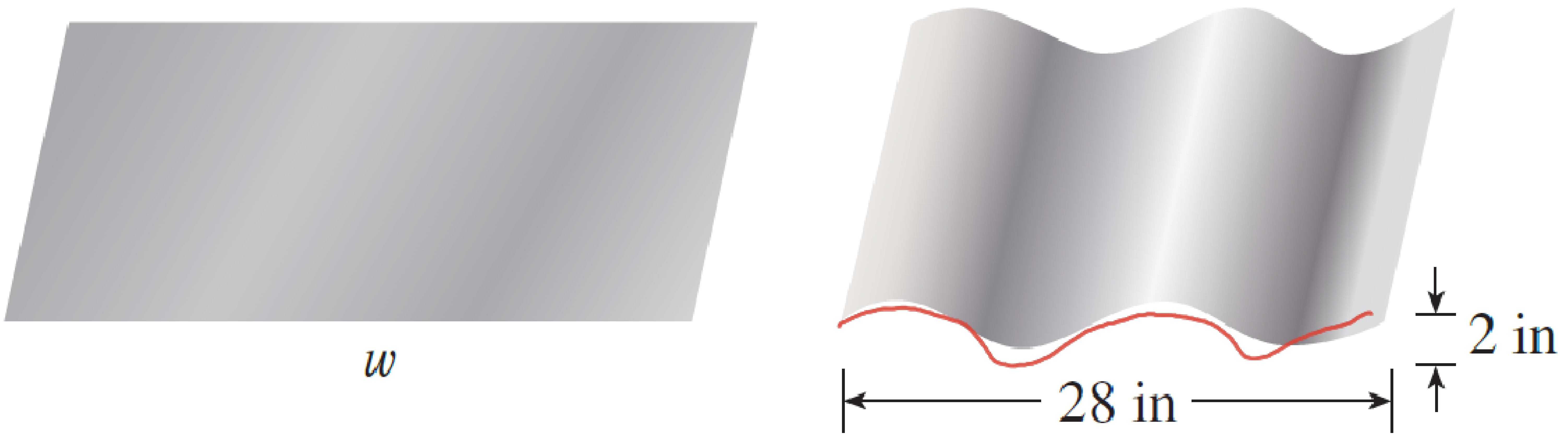
$$\frac{1}{2}\left(\frac{y^4}{4}\right) + \frac{1}{2}\left(\frac{y^{-2}}{-2}\right)\Big]_1^2$$

$$32 - 1 \quad - \quad 2 + 4$$

$$\frac{1}{8}y^4 - \frac{1}{4y^2}\Big]_1^2 = \left[\frac{16}{8} - \frac{1}{16}\right] - \left[\frac{1}{8} - \frac{1}{4}\right]$$

$$= 33/16$$

47. A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. high by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation $y = \sin(\pi x/7)$ and find the width w of a flat metal sheet that is needed to make a 28-inch panel. (Numerically evaluate the integral correct to four significant digits.)



$$y = \sin\left(\frac{\pi}{7}x\right)$$

$$\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$$

$$L = \int_0^{28} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\, dx$$

$L \approx 29.36$ inches.