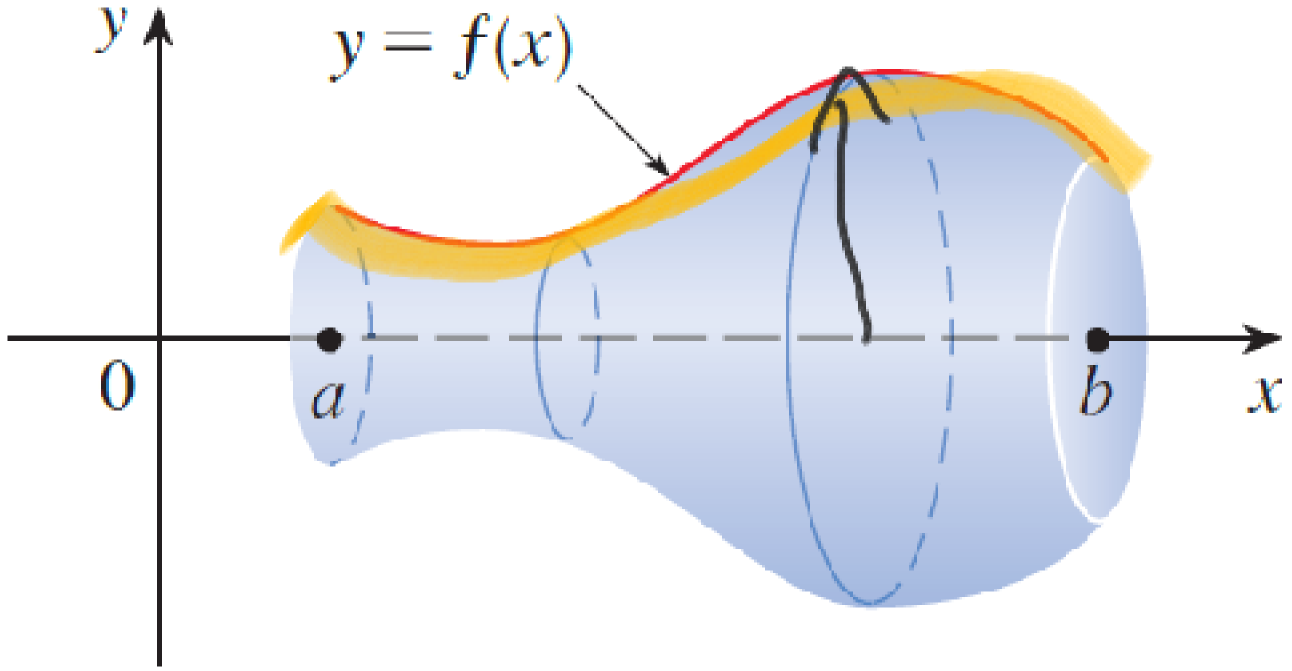
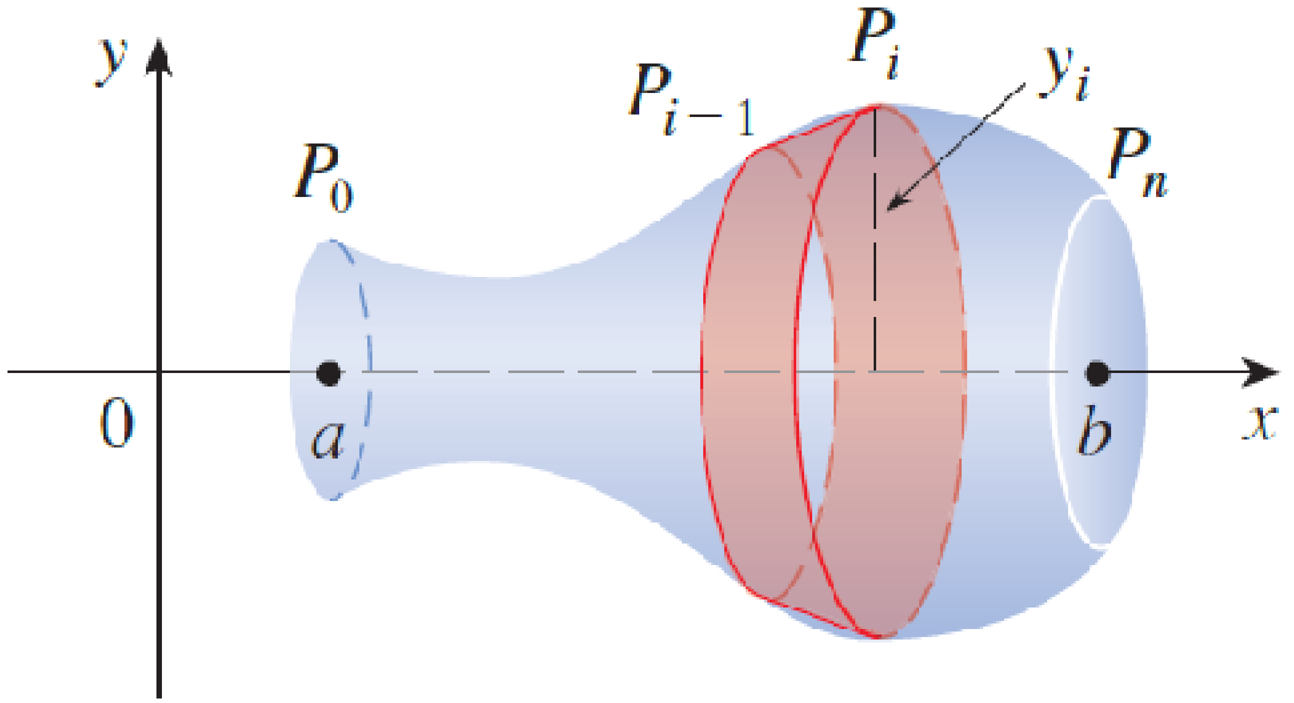
8.2 Area of a Surface of Revolution



(a) Surface of revolution



(b) Approximating band

$$SA = \underbrace{2\pi r}_{\text{circumfrance}} h$$

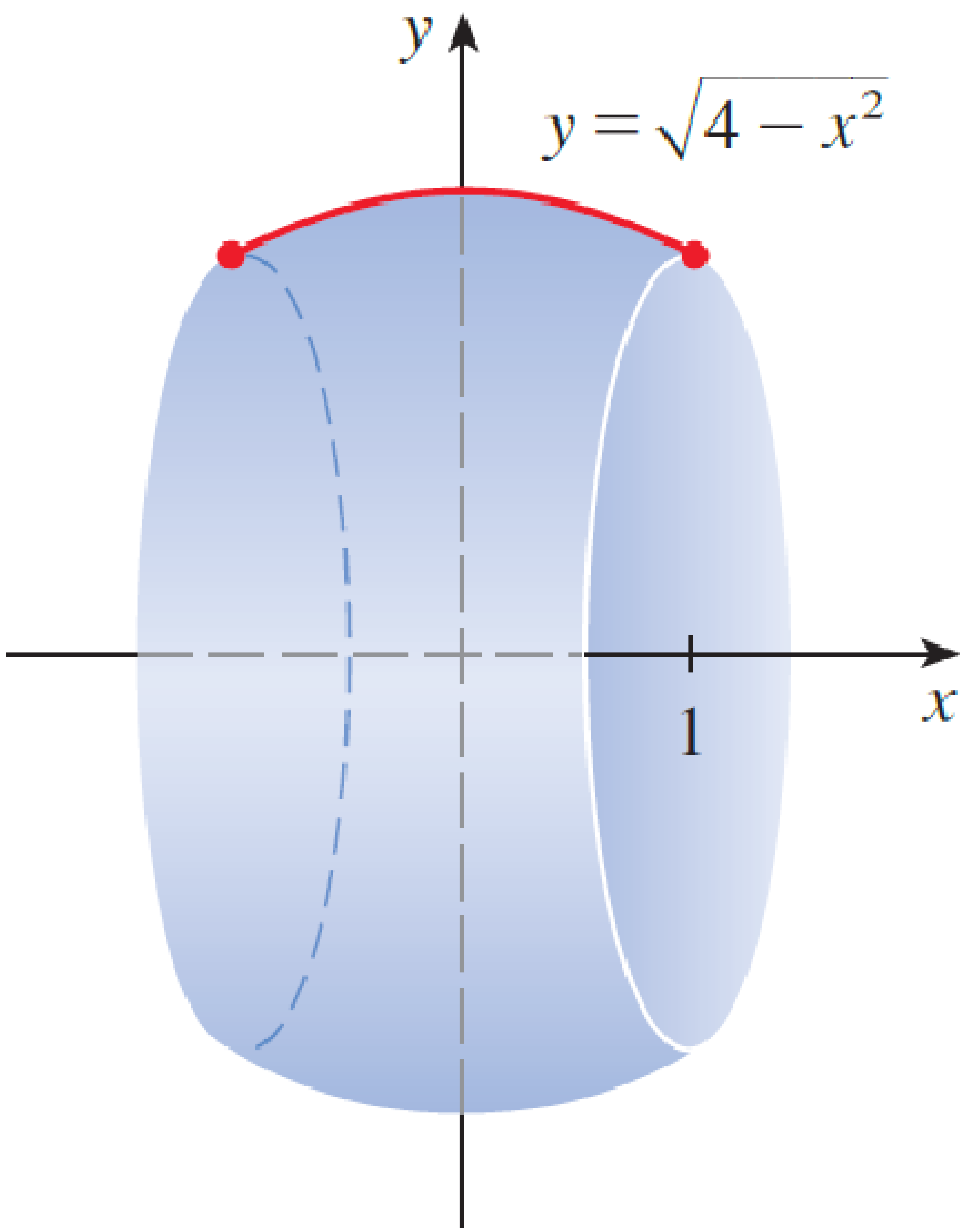
height

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

height

$y = f(x)$
radius

EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)



$$y = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4-x^2}$$

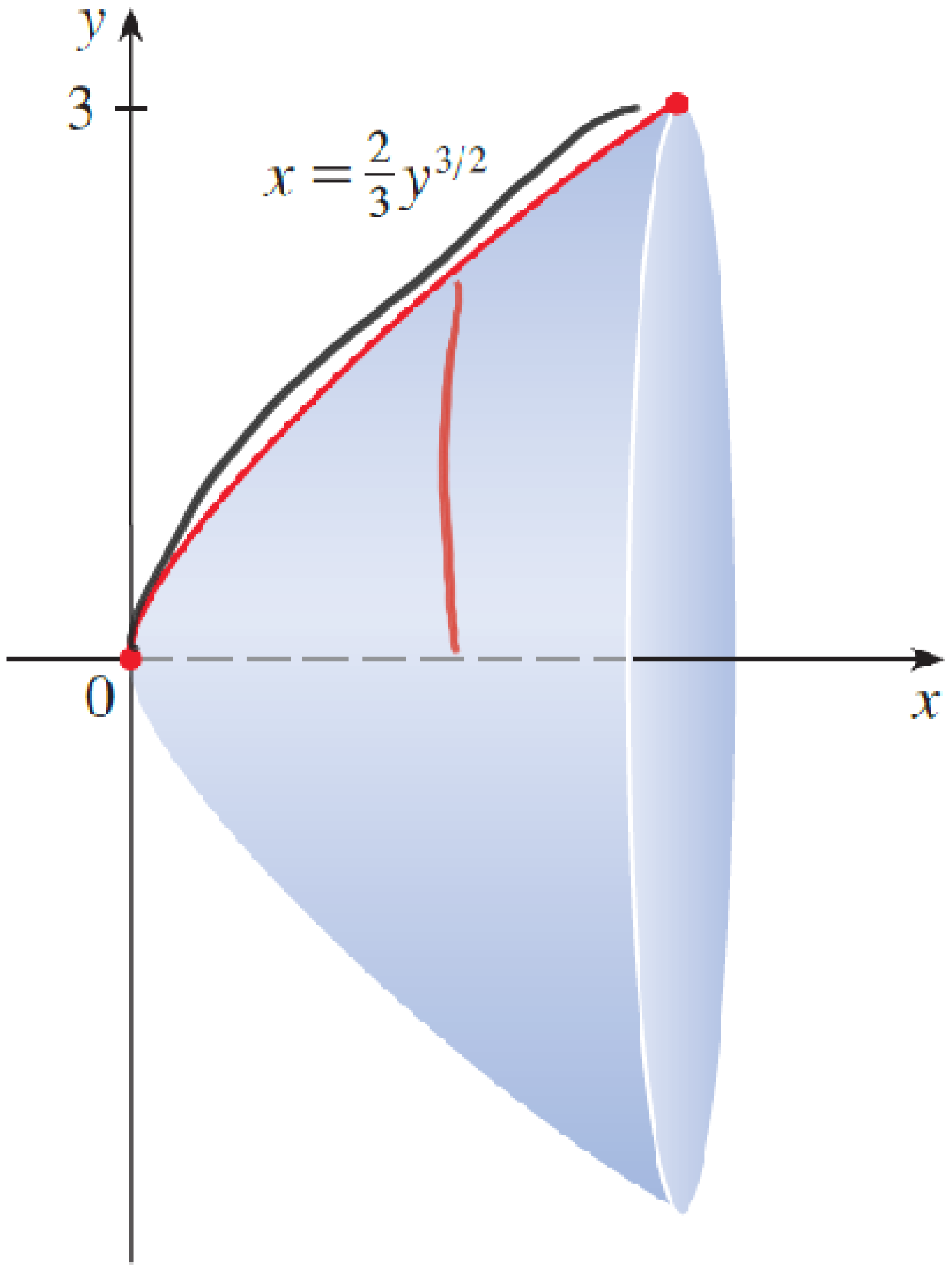
$$= \frac{4 - x^2 + x^2}{4 - x^2} = \frac{4}{4-x^2}$$

$$\sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}$$

$$2\pi \int_{-1}^{1} \left(\sqrt{4-x^2}\right)\left(\frac{2}{\sqrt{4-x^2}}\right) = 2\pi \int_{-1}^{1} 2\,dx$$

$$2\pi \Big[2x\Big]_{-1}^{1} = 2\pi\left[2+2\right] = \boxed{8\pi}$$

EXAMPLE 2 The portion of the curve $x = \frac{2}{3}y^{3/2}$ between $y = 0$ and $y = 3$ is rotated about the x-axis (see Figure 7). Find the area of the resulting surface.



$$\int_a^b 2\pi y \boxed{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\, dx$$

$$\int_a^b 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Radius

EXAMPLE 2 The portion of the curve $x = \frac{2}{3}y^{3/2}$ between $y = 0$ and $y = 3$ is rotated about the x-axis (see Figure 7). Find the area of the resulting surface.

$$x = \frac{2}{3}y^{3/2}$$

$$2\pi\int_0^3 y\sqrt{1+y}\,dy$$

$u = 1 + y$

$du = dy$

$u - 1 = y$

$$\frac{dx}{dy} = y^{1/2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + y$$

$$2\pi\int_1^4 (u-1)\sqrt{u}\,du$$

$$2\pi\int_1^4 (u-1)\sqrt{u}\,du = 2\pi\int_1^4 u^{3/2} - u^{1/2}\,du$$

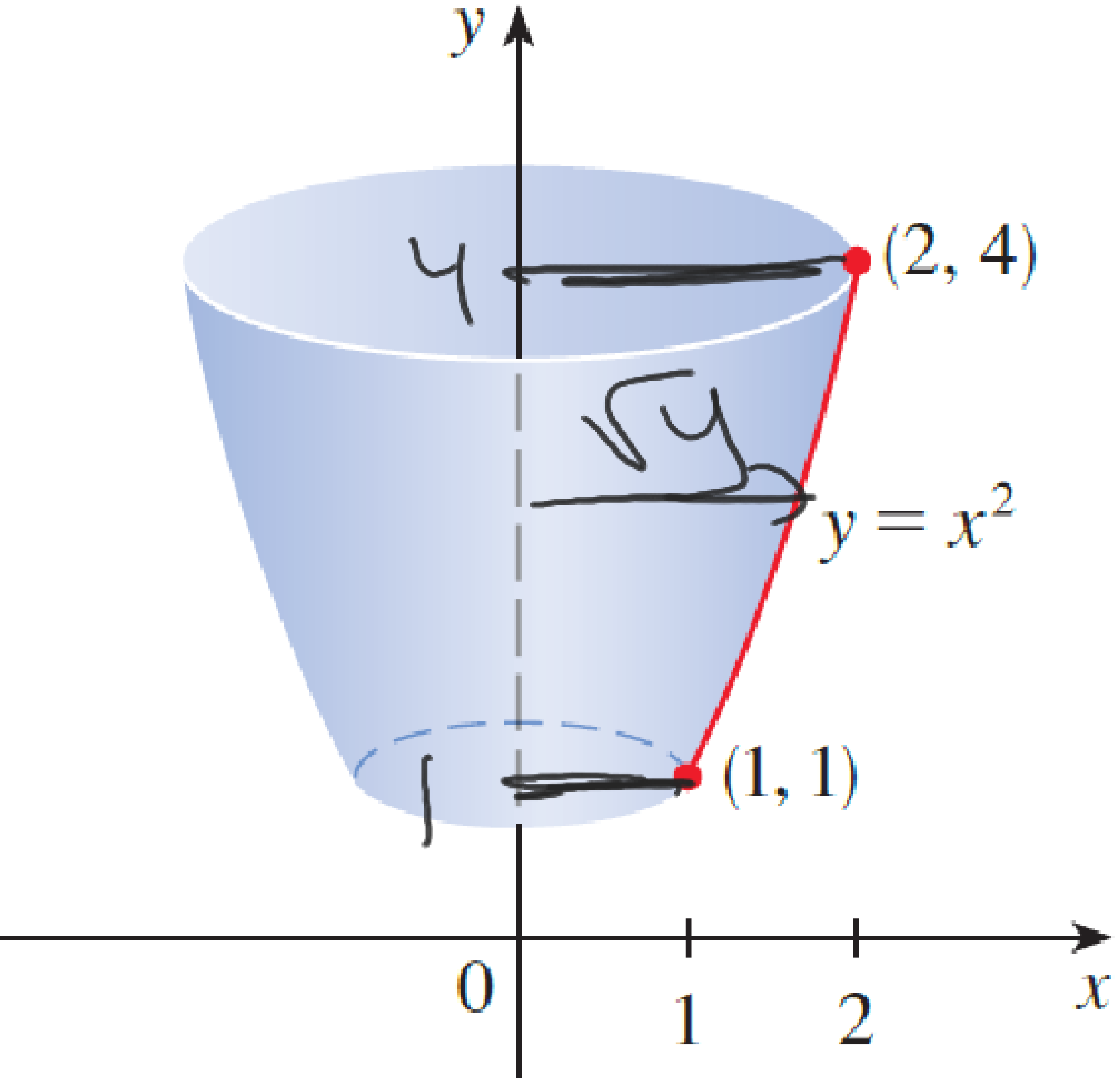
$$2\pi\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4$$

$$2\pi\left[\left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{232}{15}\pi$$

EXAMPLE 3 The arc of the parabola $y = x^2$ from (1, 1) to (2, 4) is rotated about the y-axis. Find the area of the resulting surface.

$$x = \sqrt{y} = y^{1/2}$$

$$dx/dy = \frac{1}{2\sqrt{y}}$$



$$y = x^2$$

$$dy/dx = 2x$$

$$\sqrt{1+dx/dy^2}$$

$$\sqrt{1+\frac{1}{4y}}$$

$$\sqrt{1+\left(dy/dx\right)^2}$$

$$\sqrt{1+4x^2}$$

$$2\pi\int_1^4 \sqrt{y}\left(\sqrt{1+\frac{1}{4y}}\right)dy$$

$$2\pi\int_1^4 \sqrt{y+\frac{1}{4}}\,dy$$

$$2\pi\int_1^2 x\sqrt{1+4x^2}\,dx$$

$$\sqrt{y\left(1+\frac{1}{4y}\right)} \quad y+\cancel{\frac{y}{4y}}$$

$$2\pi \int_1^4 \sqrt{y}\,\sqrt{1+\frac{1}{4y}}\,dy = 2\pi \int_1^4 \sqrt{y+\tfrac{1}{4}}\,dy$$

$$2\pi \int_1^4 \sqrt{\tfrac{1}{4}(4y+1)}\,dy = \pi \int_1^4 \sqrt{4y+1}\,dy$$

$$\sqrt{1/4} = 1/2\,(2\pi)$$

$$\frac{\pi}{4}\int_5^{17} \sqrt{u}\,du \qquad \text{(where } u = 1+4y\text{)}$$

$$\pi\,(1/4)$$

$$\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \qquad \text{(as in Solution 1)}$$

$$= 2\pi \int_1^2 x\sqrt{1+4x^2}\,dx$$

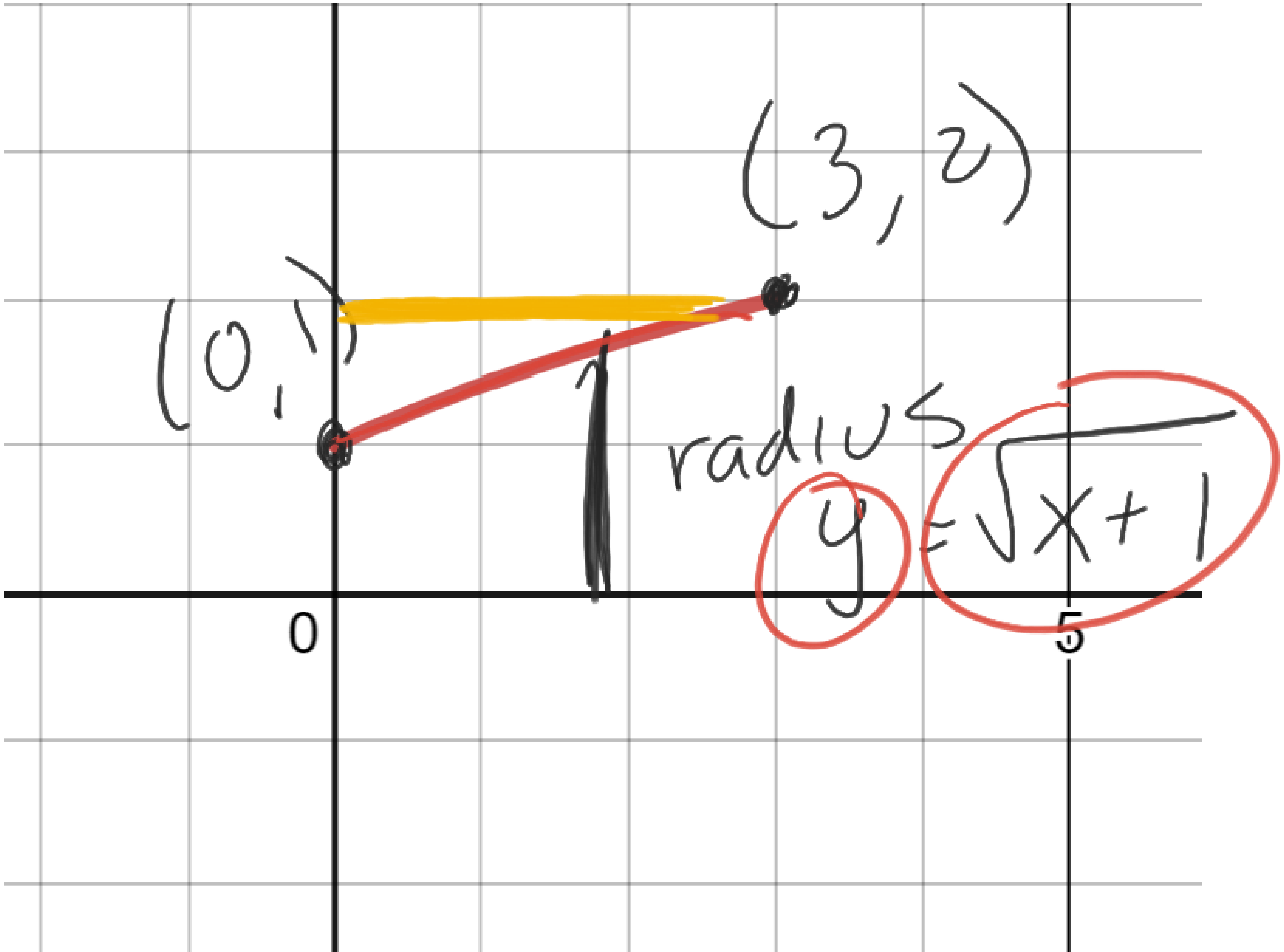
$$S = 2\pi \int_5^{17} \sqrt{u}\cdot \tfrac{1}{8}\,du$$

$$= \frac{\pi}{4}\int_5^{17} u^{1/2}\,du = \frac{\pi}{4}\left[\tfrac{2}{3}u^{3/2}\right]_5^{17}$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

11. $y^2 = x + 1$, $\boxed{0 \leqslant x \leqslant 3}$ X-axis

$x = y^2 - 1$



$$\Rightarrow \quad y = \sqrt{x+1} \quad (\text{for } 0 \le x \le 3 \text{ and } 1 \le y \le 2) \quad \Rightarrow \quad y' = 1/(2\sqrt{x+1}\,). \text{ So}$$

$$S = \int_0^3 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi \int_0^3 \boxed{\sqrt{x+1}}\sqrt{1+\frac{1}{4(x+1)}}\,dx = 2\pi \int_0^3 \sqrt{x+1+\tfrac{1}{4}}\,dx$$

$$= 2\pi \int_0^3 \sqrt{x+\tfrac{5}{4}}\,dx = 2\pi \int_{5/4}^{17/4} \sqrt{u}\,du \qquad \begin{bmatrix} u = x + \frac{5}{4}, \\ du = dx \end{bmatrix}$$

$$= 2\pi \left[\tfrac{2}{3}u^{3/2}\right]_{5/4}^{17/4} = 2\pi \cdot \frac{2}{3}\left(\frac{17^{3/2}}{8} - \frac{5^{3/2}}{8}\right) = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).$$

$$2\pi \int_1^2 y^2 \sqrt{1+4y^2}^{\,1} - \sqrt{(1+4y^2)}$$

$$\boxed{y^2 - 1 = x}$$

$$dx/dy = 2y$$

$$\sqrt{1 + (\ )^2} = \sqrt{1 + 4y^2}$$

$$2\pi \int_1^2 y \sqrt{1 + 4y^2}\, dy$$

20. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x,\ 1 \leqslant x \leqslant 2$ y-axis

$$\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x} = \frac{x}{2} - \frac{1}{2x}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4x^2}$$

$$= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$\sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} = \frac{x}{2} + \frac{1}{2x}$$

$$2\pi\int_1^2 x\left(\frac{x}{2}+\frac{1}{2x}\right)dx$$

$$2\pi\int_1^2 \frac{x^2}{2}+\frac{1}{2}\,dx = 2\pi\left[\frac{x^3}{6}+\frac{x}{2}\right]_1^2$$

$$=2\pi\left(\frac{8}{6}+\frac{2}{2}-\frac{1}{6}-\frac{1}{2}\right)=\boxed{\frac{10}{3}\pi}$$

about the y-axis. If the dish is to have a 10-ft diameter and a maximum depth of 2 ft, find the value of a and the surface area of the dish.

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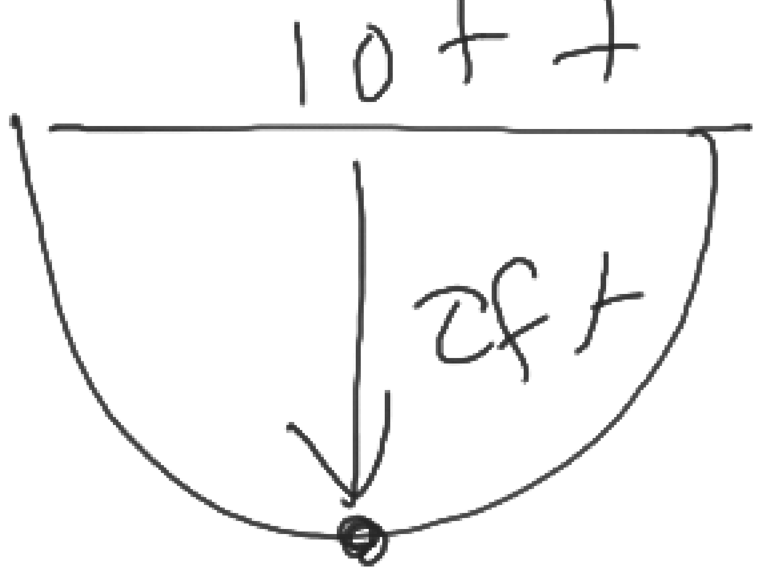
$y = ax^2$



$ac^2 = 2$

$2c = 10$ | $a(25) = 2$

$c = 5$ | $a = \frac{2}{25}$

$y = \frac{2}{25}x^2$

y-axis

$0 < x < 5$

$0 < y < 2$

$$2\pi \int_0^5 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx$$

$\frac{4x}{25} = \frac{16x^2}{625}$

$1 + \frac{16x^2}{625}$

$\sqrt{\frac{625 + 16x^2}{625}}$

$\sqrt{625}$

25

$$\frac{2\pi}{25} \int_0^5 x \sqrt{625 + 16x^2}\, dx$$

$$\frac{2\pi}{25}\int_0^5 x\sqrt{625+16x^2}\,dx$$

$$u = 625 + 16x^2$$

$$du = 32x\,dx$$

$$\frac{1}{32}\,du = x\,dx$$

$$\frac{\pi}{400}\int_0^5 \sqrt{u}\,du \qquad \frac{2}{3}u^{3/2}$$

$$\frac{\pi}{600}\left[(625+16x^2)^{3/2}\right]_0^5$$