9.3 Separable Equations

$$\boxed{\frac{dy}{dx} = g(x)f(y)}$$

$$\int 6y\,dy = \int 8x\,dx$$

$$\frac{6y^2}{2} \qquad \frac{8x^2}{2}$$

$$3y^2 + C_1 = 4x^2 + C_2$$

$C_1 + C_2$

$$3y^2 = 4x^2 + 2$$

$$\frac{d}{dx}(3y^2) = \frac{d}{dx}(4x^2 + 2)$$

$$6y\left(\frac{dy}{dx}\right) = 8x$$

$$\frac{dy}{dx} = \frac{8x}{6y}$$

EXAMPLE 1

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

$$\frac{dy}{dx} = \frac{x^2}{y^2} \longrightarrow \int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y^3 = x^3 + 3C$$

$$y = \sqrt[3]{x^3 + C}$$

EXAMPLE 1

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

$x \quad y$

$y(0) = 2$

$\frac{dy}{dx} = \frac{x^2}{y^2} \rightarrow y = \sqrt[3]{x^3 + C}$

$2 = \sqrt[3]{C} \rightarrow C = 8$

$\boxed{y = \sqrt[3]{x^3 + 8}}$

EXAMPLE 2 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$

$$\int (2y + \cos y)\, dy = \int 6x^2\, dx$$

$$\frac{2y^2}{2} \qquad \frac{6x^3}{3}$$

$$\boxed{y^2 + \sin y = 2x^3 + C}$$

EXAMPLE 3 Solve the differential equation $y' = x^2 y$

$$y' = \frac{dy}{dx}$$

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{y} = \frac{x^2 y}{y}\,dx$$

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$e^{\ln|y|} = e^{\left(\frac{1}{3}x^3 + C\right)}$$

$$|y| = e^{\left(\frac{1}{3}x^3 + C\right)}$$

$$y = \boxed{\pm e^{C}} \cdot e^{\frac{1}{3}x^3}$$

$$y = A e^{x^3/3}$$

19. $x \ln x = y(1 + \sqrt{3 + y^2})\, y', \quad y(1) = 1$

$$x \ln x = y\left(1 + \sqrt{3+y^2}\right)\frac{dy}{dx}$$

$$\boxed{\int (x \ln x)\, dx} = \boxed{\int y\left(1+\sqrt{3+y^2}\right) dy}$$

$$\frac{x^2}{2}\ln x - \int \frac{1}{2}x\, dx \qquad \int y + y\sqrt{3+y^2}\, dy$$

$$\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C = \frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2}$$

$u = \ln x \quad v = \frac{x^2}{2}$

$du = \frac{1}{x}\, dx \quad dv = x$

$\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)$

$u = 3 + y^2$

$du = 2y$

$\frac{1}{2}du = y\, dy$

$\cancel{\frac{1}{2}}\left(\cancel{\frac{2}{3}}(3+y^2)^{3/2}\right)$

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19. $x \ln x = y\left(1 + \sqrt{3 + y^2}\right) y', \quad y(1) = 1$

$x \quad y$

$$\frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C = \frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2}$$

$$\frac{1}{2}(0) - \frac{1}{4}(1) + C = \frac{1}{2} + \frac{1}{3}(4)^{3/2}$$

$$-\frac{1}{4} + C = \frac{1}{2} + \frac{1}{3}(8)$$

$$C = \frac{1}{2} + \frac{8}{3} + \frac{1}{4} = \frac{41}{12}$$

$6 \quad 32 \quad 3$

EXAMPLE 4 In Section 9.2 we modeled the current $I(t)$ in the electric circuit shown in Figure 5 by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

$L = 4 \quad R = 12$
$E(t) = 60$

Find an expression for the current in a circuit where the resistance is 12 Ω, the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

$$4\frac{dI}{dt} + 12I = 60$$

$$\frac{dI}{dt} + 3I = 15$$

$$dI + 3I\,dt = 15\,dt$$

$$\frac{dI}{dt} = 15 - 3I$$

$$\frac{dI = (15 - 3I)\,dt}{15 - 3I}$$

$$\int \frac{1}{15 - 3I}\,dI = \int dt$$

$$-\frac{1}{3}\ln|15 - 3I| = t + C$$

$$-\tfrac{1}{3}\ln|15 - 3I| = t + C$$

$$|15 - 3I| = e^{-3(t+C)}$$

$$15 - 3I = \pm e^{-3C}e^{-3t} = Ae^{-3t}$$

$$I = 5 - \tfrac{1}{3}Ae^{-3t}$$

$I \to 5$ Amps

$y(0) = 20$

EXAMPLE 6 A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt will there be in the tank after half an hour?

$$\frac{dy}{dt} = (\text{Rate In}) - (\text{Rate out})$$

$$\text{Rate In} = \left(\frac{0.03\,\text{kg}}{\text{L}}\right)\left(\frac{25\,\text{L}}{\text{min}}\right) = \frac{0.75\,\text{kg}}{\text{min}}$$

$$\text{Rate Out} = \left(\frac{y(t)}{5000\,\text{L}}\right)\left(\frac{25\,\text{L}}{\text{min}}\right) = \frac{y(t)\,\text{kg}}{200\,\text{min}}$$

$$\frac{dy}{dt} = 0.75 - \frac{1}{200}y(t)$$

$$\frac{dy}{dt} = \frac{150 - y(t)}{200}$$

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - \overset{20}{y}| = \frac{\overset{0}{t}}{200} + C$$

$$-\ln(130) = C$$

$$-\ln|150 - y| = \frac{t}{200} - \ln(130)$$

$$-\ln|150-y| = \frac{t}{200} - \ln(130) \qquad t = 30$$

$$-\ln|150-y| = \frac{30}{200} - \ln(130)$$

$$e^{\ln|150-y|} = e^{4.718}$$

$$150 - y = \pm e^{4.718}$$

$$-y = \pm e^{4.718} - 150$$

$$y = \pm e^{4.718} + 150$$

$$\cancel{y = 26.9}$$

$$\boxed{y = 38.1}$$

49. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?