Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies the conditions

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

$$\sum_{n=1}^{\infty} \cancel{(-1)^{n-1}} \boxed{b_n}$$

The alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$$b_n=\frac{1}{n}$$

AST

1) alternating $\rightarrow$ yes

2) $\lim_{n\to\infty} b_n = 0 \rightarrow$ yes

3) $b_{n+1} \leq b_n \rightarrow$ yes

$$\frac{1}{n+1} \leq \frac{1}{n}$$

$\boxed{\text{Converges}}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$$

$$b_n = \frac{3n}{4n-1}$$

$\boxed{\text{Dverges T.f.D.}}$

1) alternating → yes

→ 2) $\lim_{n\to\infty} b_n = 0$ → NO $\lim_{n\to\infty} \frac{3n/n}{4n/n - 1/n} = \frac{3}{4}$

$\boxed{3)\ b_{n+1} \leq b_n}$ X

Test for Divergance

$\lim_{n\to\infty} b_n \neq 0$

Test the series $\sum_{n=1} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

$$b_n = \frac{n^2}{n^3+1}$$

$$\lim_{n \to \infty} \frac{n^2/n^3}{n^3/n^3 + 1/n^3} = \frac{0}{1} = 0$$

$$b_{n+1} \le b_n$$

$$\frac{(n+1)^2}{(n+1)^3+1} \le \frac{n^2}{n^3+1}$$

Converges

$$b_n = \frac{n^2}{n^3+1}$$

$$\frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

$f'(x) = 2x$

$g'(x) = 3x^2$

$$\boxed{\frac{(2x)(x^3+1) - (3x^2)(x^2)}{(x^3+1)^2}} < 0$$

$2x^4 + 2x - 3x^4$

$2x - x^4 < 0$

$x(2 - x^3) < 0$

$\boxed{x < 0}$

$2 - x^3 < 0$

$-x^3 < -2$

$x^3 > 2$

$\boxed{x > \sqrt[3]{2}}$

A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

We have discussed convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 7 that the idea of absolute convergence sometimes helps in such cases.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$P = 2 > 1$ Converges

abslutely convergent

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| \to \sum_{n-1}^{\infty} \frac{1}{n} \quad \text{Divergent}$$

Convergent

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$ absoluctly convergent

ent.

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

Converges

converges $p = 2 > 1$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \boxed{\sin \frac{\pi}{n}}$

Sin is + $0 \to \pi$

$0 \leq \frac{\pi}{n} \leq \pi$

1) Alternating ✓

2) $\lim_{n\to\infty} b_n = 0$ ✓

$\lim_{n\to\infty} \sin\frac{\pi}{n} \to 0$

$\sin \lim_{n\to\infty} \frac{\pi}{n}$

$\sin 0 = 0$

3) $b_{n+1} \leq b_n$

$\sin\frac{\pi}{n+1} \leq \sin\frac{\pi}{n}$ ✓

$\boxed{\text{Converges AST}}$

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \frac{\left(\frac{(n+1)^3}{3^{n+1}}\right)}{\frac{n^3}{3^n}} \rightarrow \frac{1}{3}$$

$$\frac{(n+1)^3}{3^{n+1}} \cdot \frac{\cancel{3^n}}{n^3} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \boxed{\frac{1}{3}}$$

$$3^{n+1} = \cancel{(3^n)}(3)$$

$$\left(1+\frac{1}{n}\right)^3$$

of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$. Ratio

Diverges

$$\frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}\cancel{(n!)}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e > 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Ratio

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n/n}{n+1/n/n} = \frac{1}{1+0} = \boxed{1}$$

IDK

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ Converges $p = 2 > 1$

p - series

$$\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \left(\frac{n/n}{n+1/n/n}\right)^2 = \left(\frac{1}{1+0}\right)^2 = \boxed{1}$$

IDK

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

$\sqrt[n]{|a_n|}$

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

he series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$\sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \frac{2n+3}{3n+2}$$ ← Root Test

$$\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

absolutly Convergent

s $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n}$ converges or diverges.

$$\sqrt[n]{\left|\left(\frac{n+1}{n}\right)^{n}\right|} = \frac{n+1}{n}$$

Test for Divergence

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n}$$

$$\lim_{n\to\infty} \frac{\cancel{n/n}+1/n}{n/n} = 1$$

$$\frac{1+0}{1}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$$