10.1 Curves Defined by Parametric Equations

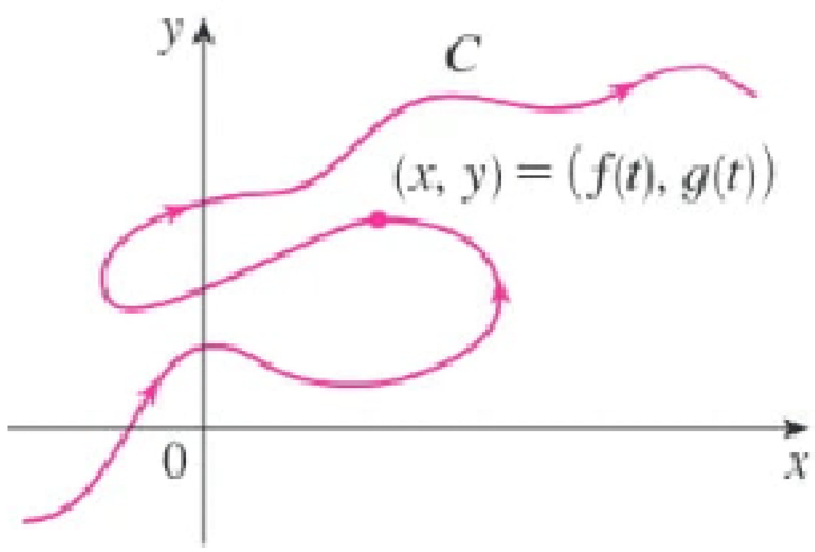


FIGURE 1

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve.** The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and therefore we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5

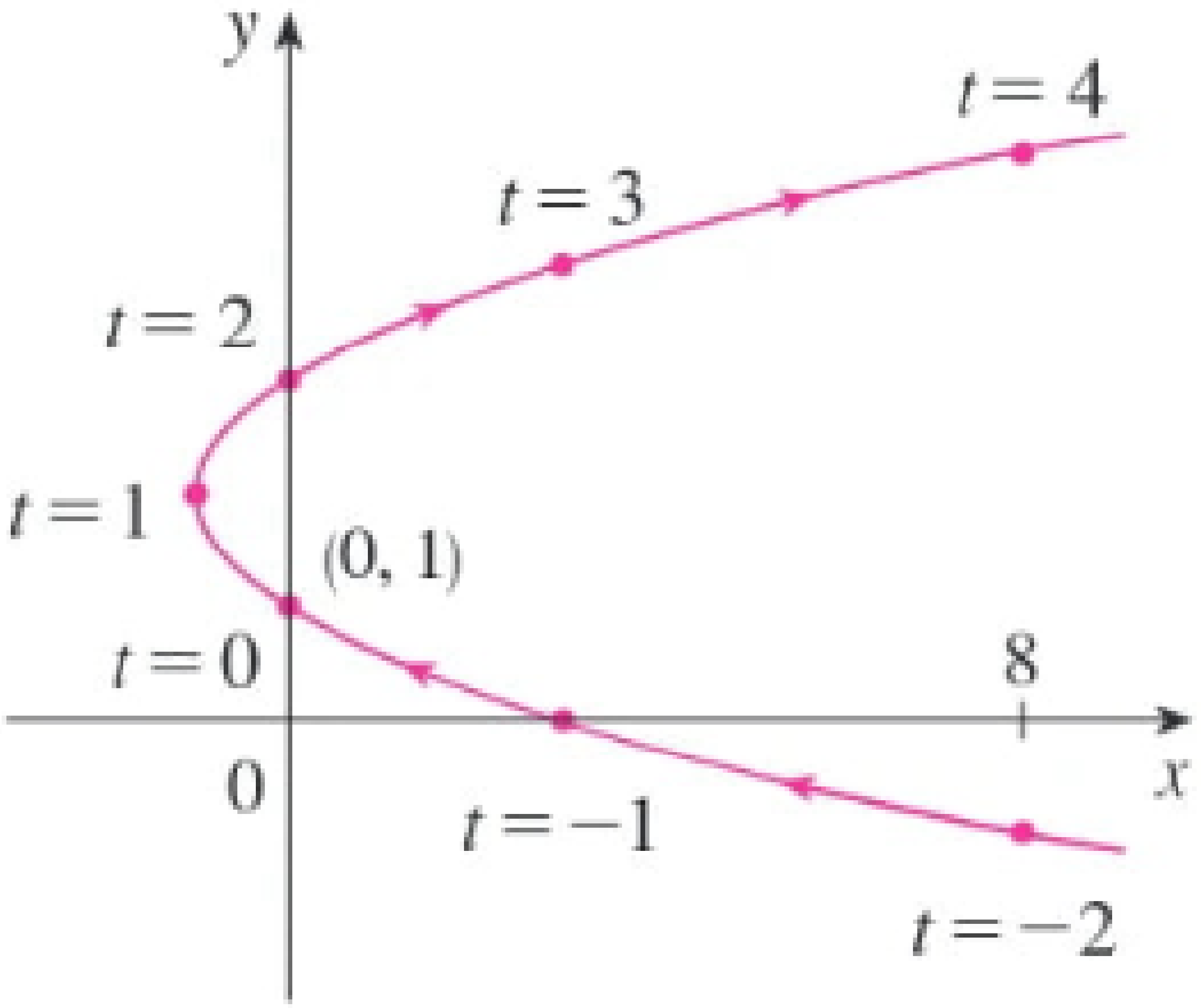


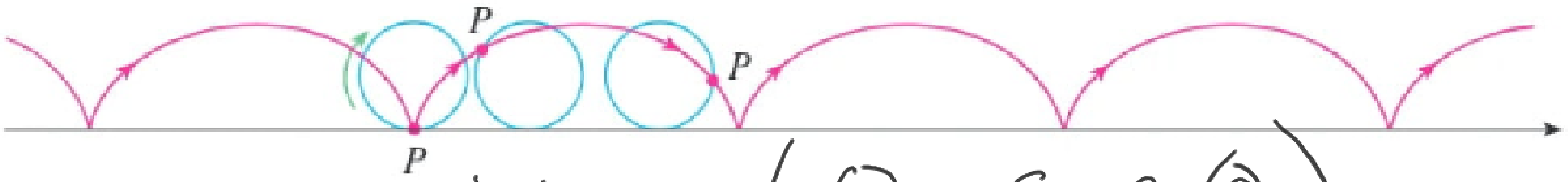
FIGURE 2

$x = \sin t$

$y = \cos t$

t	x	y
$-\pi/2$		
$-\pi/4$		
0		
$\pi/4$		
$\pi/2$		

EXAMPLE 7 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 13). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



$$x = r(\theta - \sin\theta)$$

$$y = r(1 - \cos\theta)$$

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

1 $$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point (3, 0) and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

$x = t^2 \qquad y = t^3 - 3t$

$\sqrt{3,3,3} \qquad \sqrt{27}$

a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$

$\sqrt{3} = \sqrt{t^2} \Rightarrow t = \boxed{\pm\sqrt{3}}$

$(\sqrt{3})^3 + 3\sqrt{3}$

$3\sqrt{3}$

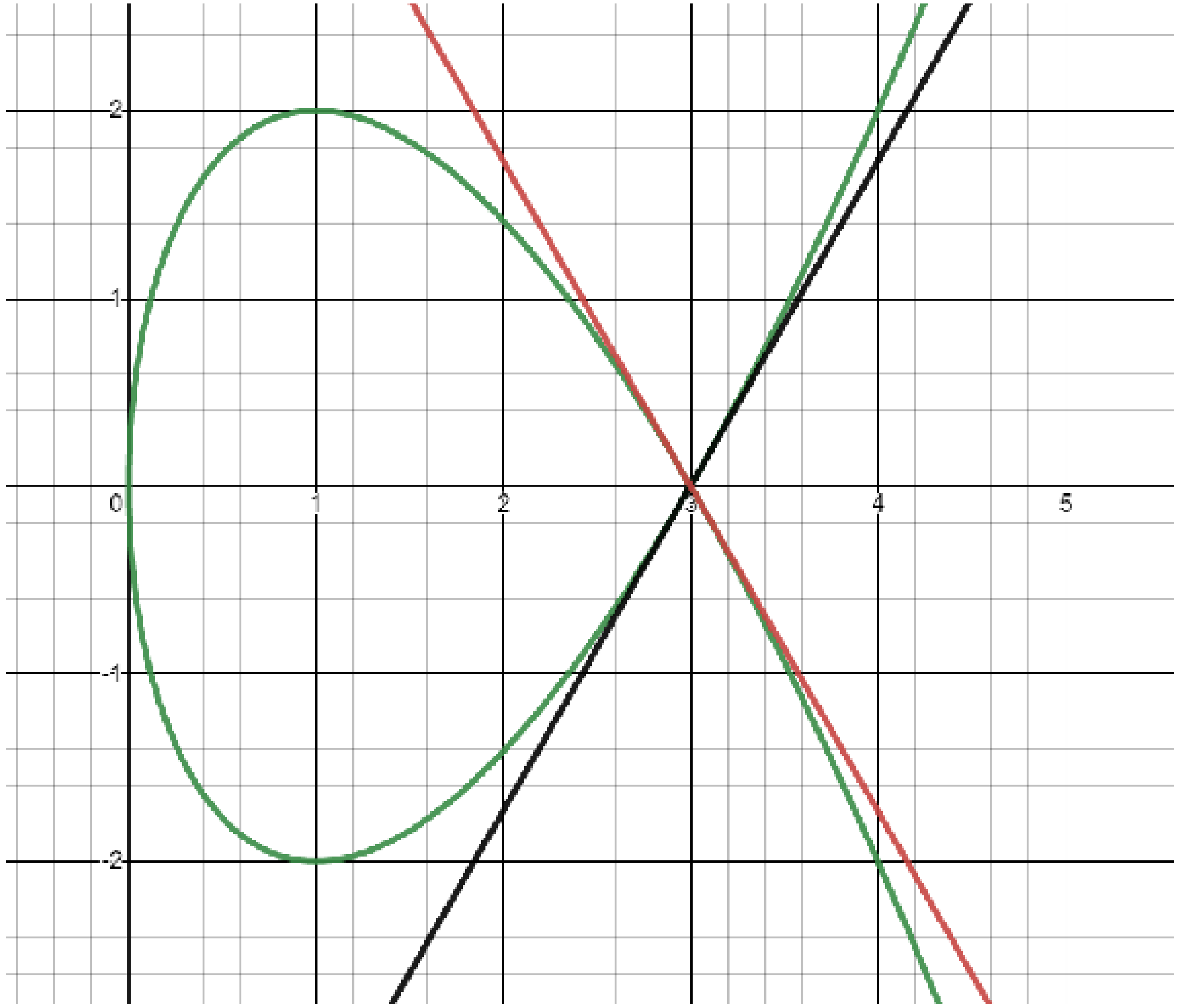
$t = \pm\sqrt{3} \quad (x, y) = (3, 0)$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \pm\sqrt{3}(x - 3)$$

$$\boxed{y = \pm\sqrt{3}(x - 3)}$$



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$x = t^2$ $\qquad$ $y = t^3 - 3t$

b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$

$3t^2 - 3 = 0$ $\qquad$ $2t = 0$

Horz $\qquad$ Vert

$3t^2 - 3 = 0$

$3t^2 = 3$

$\sqrt{t^2} = \sqrt{1}$

$t = \pm 1$ Horz

$x = 1^2 = 1$ $x = (-1)^2 = 1$

$y = (1)^3 - 3(1)$
$= -2$

$y = (-1)^3 - 3(-1)$
$= 2$

$(1, -2)$ $(1, 2)$

$2t = 0$

$t = 0$ Vert

$x = t^2$ $y = t^3 - 3t$

$x = 0$ $y = 0$

$(0, 0)$

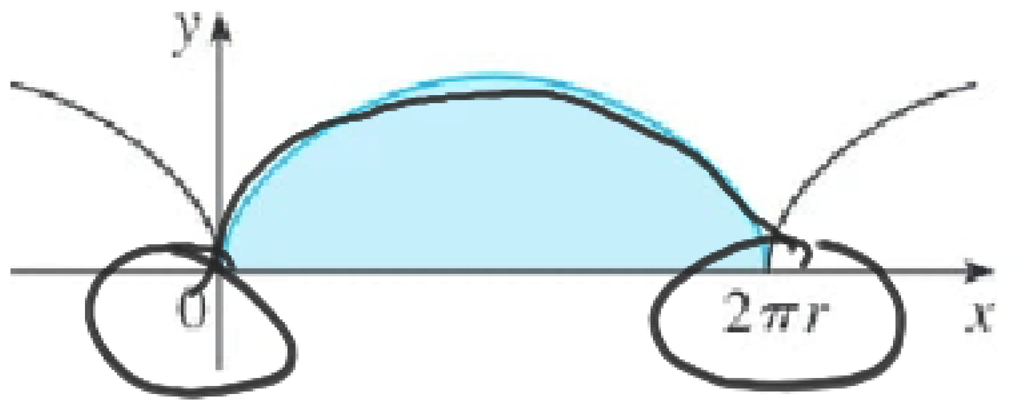
Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

EXAMPLE 3 Find the area under one arch of the cycloid

$$\overset{f(t)}{x} = \boxed{r(\theta - \sin\theta)} \qquad \overset{g(t)}{y} = r(1 - \cos\theta)$$



$$\int_a^b g(t)\, f'(t)\, dt$$

$$f'(\theta) = r(1 - \cos\theta)$$

$$\int_0^{2\pi r} [r(1-\cos\theta)][r(1-\cos\theta)]\, d\theta$$

$$\int_0^{2\pi r}\left[r(1-\cos\theta)\right]\left[r(1-\cos\theta)\right]d\theta$$

$$= r^2\int_0^{2\pi}\frac{3}{2}-2\cos\theta+\tfrac{1}{2}\cos 2\theta$$

$$r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$r^2\int_0^{2\pi}1-2\cos\theta\ \boxed{+\cos^2\theta}\ d\theta$$

$$+\tfrac{1}{2}(1+\cos 2\theta)$$

$$+\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta$$

$$= r^2 \int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta$$

$$r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta \right]_0^{2\pi}$$

$$r^2 \left(\frac{3}{2}(2\pi) \cancel{-2\sin 2\pi} + \cancel{\tfrac{1}{4}\sin 4\pi} \right) - \left(\cancel{\tfrac{3}{2}(0)} - \cancel{2(\sin 0)} + \cancel{\tfrac{1}{4}\sin 0} \right)$$

$$r^2(3\pi) = \boxed{3\pi r^2}$$

m α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting For
tution Rule, we obtain

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}$$

ave

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

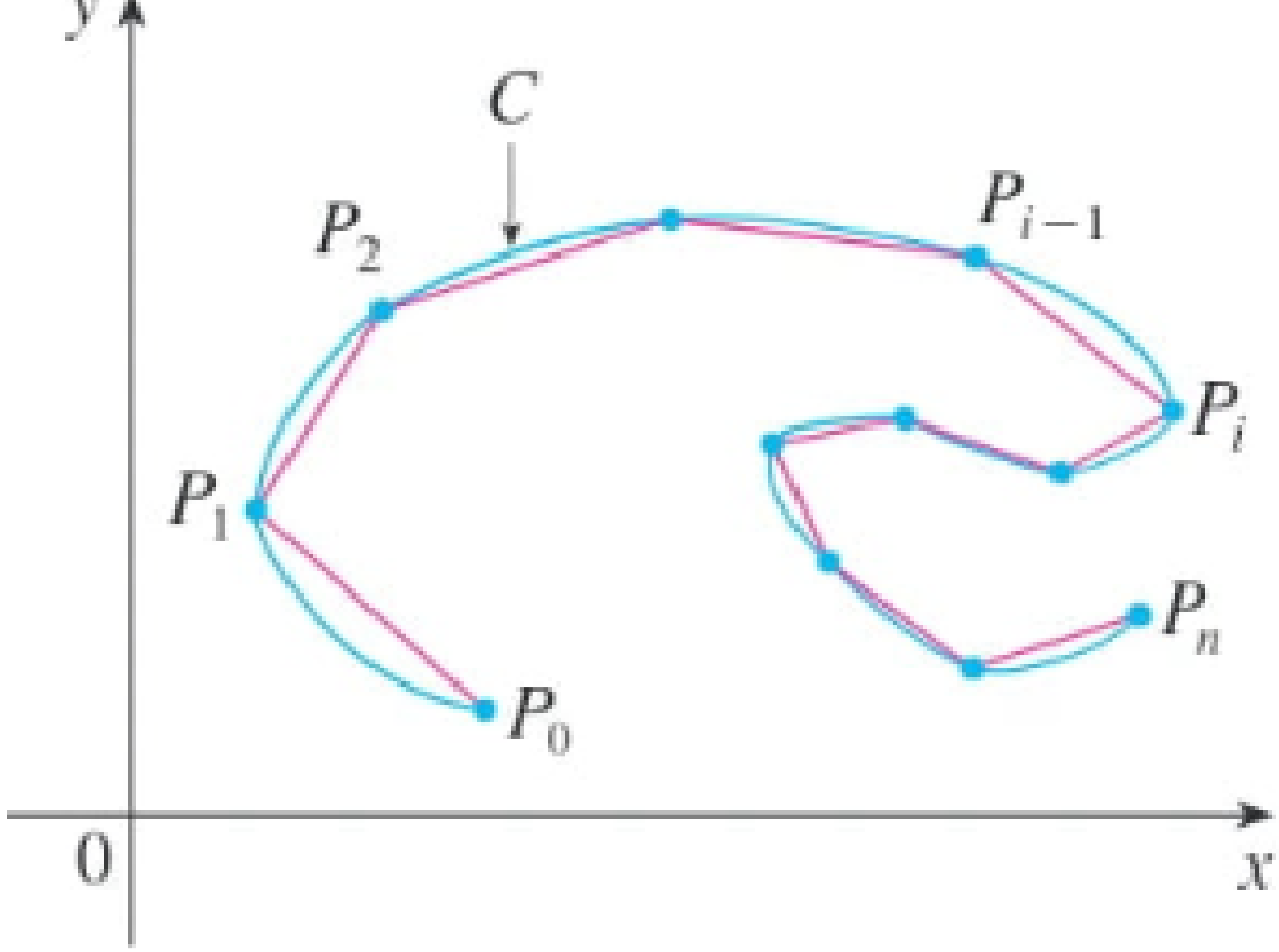
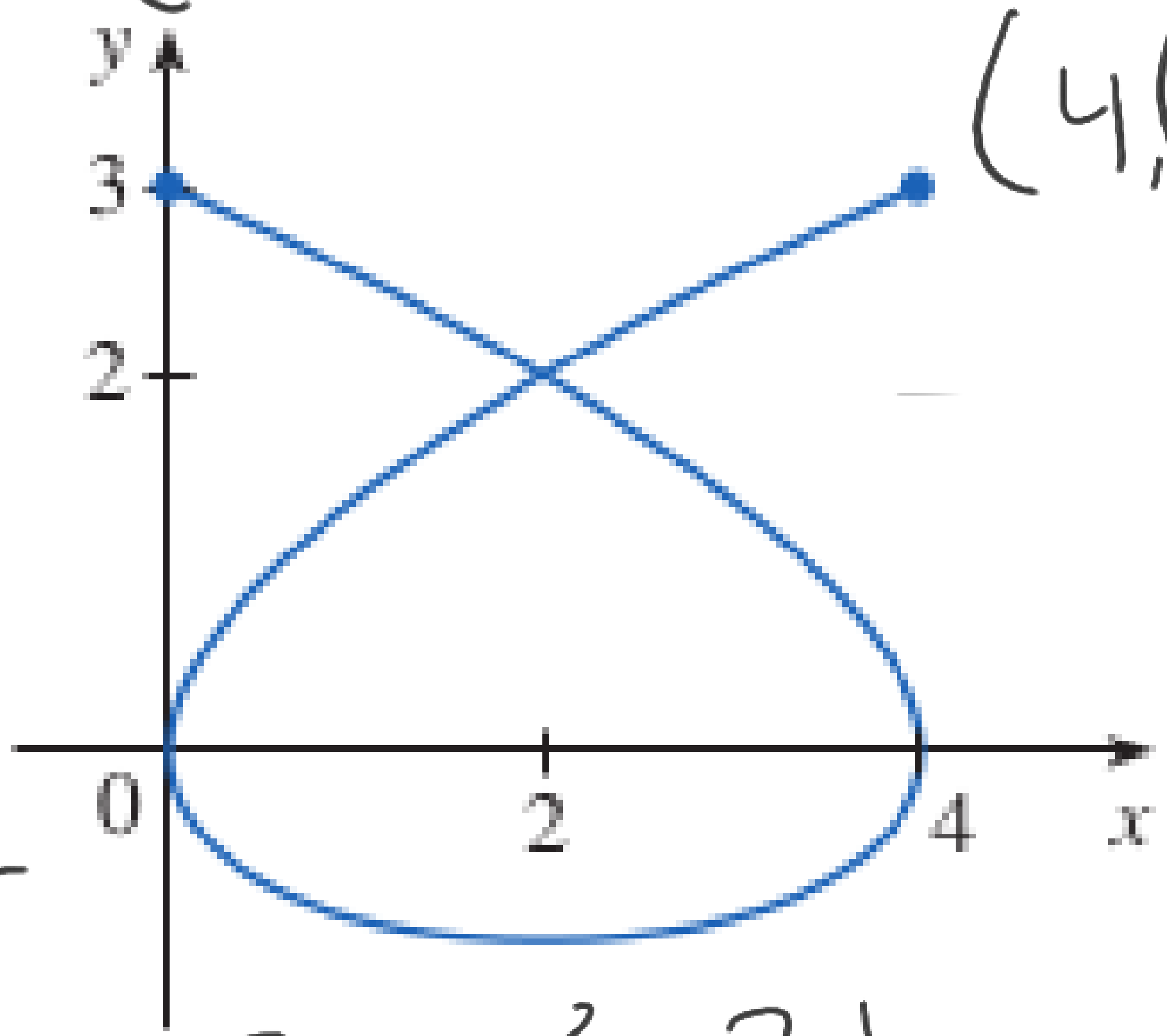


FIGURE 4

43. $x = 3t^2 - t^3, \quad y = t^2 - 2t$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(0, 3)



(4, 3)

$x' = 6t - 3t^2$

$y' = 2t - 2$

$$\int_{-1}^{3} \sqrt{(6t - 3t^2)^2 + (2t - 2)^2}$$

$3 = t^2 - 2t$

$0 = t^2 - 2t - 3$

$(t-3)(t+1)$

$$\int_{-1}^{3} \sqrt{(6t-3t^2)^2 + (2t-2)^2}$$

$$(36t^2 - 36t^3 + 9t^4) + (4t^2 - 8t + 4)$$

$$9t^4 - 36t^3 + 40t^2 - 8t + 4$$

$$\int_{-1}^{3} \sqrt{} \approx 15.21$$

Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. Suppose the curve c given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f', g' are continuous, $g(t) \geqslant 0$, is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by

6 $$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

radius

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, but for parametric curves we use

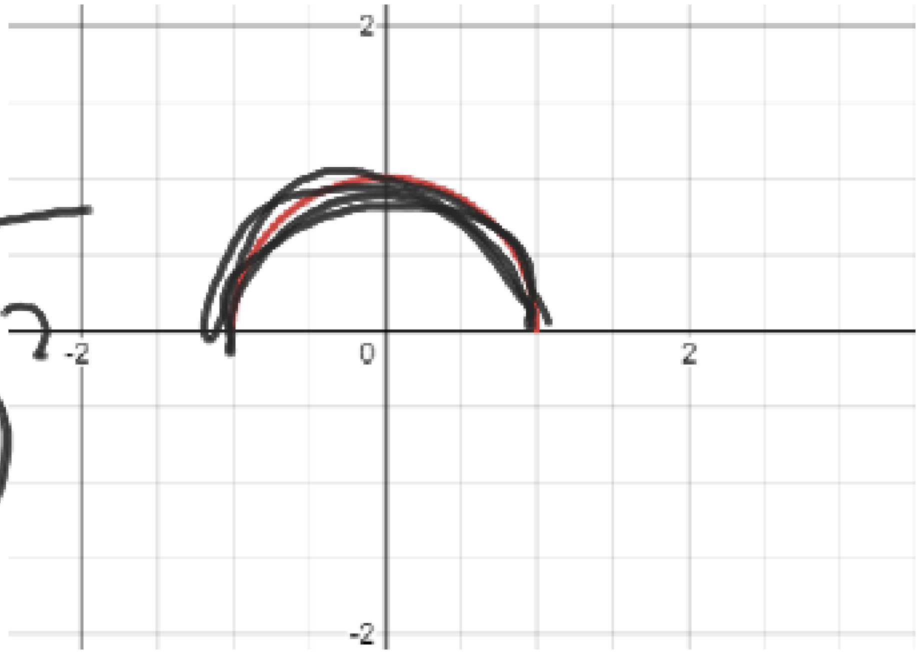
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 6 Show that the surface area of a sphere of radius r is $4\pi r^2$.

$4\pi r^2$

$x = r\sin\theta \qquad y = r\cos\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi\,(r\cos\theta)\sqrt{(r\cos\theta)^2 + (-r\sin\theta)^2}$$



$$2\pi r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$r^2\boxed{(\cos^2\theta + \sin^2\theta)} = 1$$

$$2\pi r \int_{-\pi/2}^{\pi/2} r\cos\theta \, d\theta$$

$$2\pi r \int_{-\frac{\pi}{2}}^{\pi/2} r\cos\theta \, d\theta$$

$$4\pi r^2 \int_0^{\pi/2} \cos\theta \, d\theta$$

$$4\pi r^2 \left[\sin\theta\right]_0^{\pi/2}$$

$$4\pi r^2 (1 - 0) = \boxed{4\pi r^2}$$