1. Test for Divergence If you can see that $\lim_{n\to\infty} a_n$ may be different from 0, then apply the Test for Divergence.

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$a_n = \frac{1}{n^2} \to 0$

C

can't use this test

$a_n = \frac{n^2}{5}$

D

2. **p -Series** If the series is of the form $\sum 1/n^p$, then it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

3. **Geometric Series** If the series has the form $\boxed{\sum ar^{n-1} \text{ or } \sum ar^n}$, then it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

$\frac{1}{n^2}$ $2>1$

C

$n^3 = \frac{1}{n^{-3}}$ $-3 \leq 1$ D

4. **Comparison Tests** If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply a comparison test to $\sum|a_n|$ and test for absolute convergence.

The Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

5. Alternating Series Test If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility. Note that if $\sum b_n$ converges, then the given series is absolutely convergent and therefore convergent.

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies the conditions

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

6. Ratio Test Series that involve factorials or other products (including a constant raised to the n th power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n . Thus the Ratio Test should not be used for such series.

The Ratio Test

(i) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

7. Root Test If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

8. Integral Test If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^\infty a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

18. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Integral Test

Converges

$f(x) = x^2 e^{-x^3}$

$u = -x^3$

$du = -3x^2\,dx$

$-\frac{1}{3}du = x^2\,dx$

$\displaystyle\int_1^\infty -\frac{1}{3}e^u\,du = -\frac{1}{3}e^u = -\frac{1}{3}e^{-x^3}\Big]_1^\infty$

$= -\frac{1}{3}\cdot\frac{1}{e^{x^3}} = \frac{1}{3e}$

23. $$\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$$

$$\rightarrow \frac{2^k \cdot 2^{-1} \cdot 3^k \cdot 3}{k^k}$$

$$\rightarrow \frac{3}{2}\boxed{\left(\frac{6}{k}\right)^k}$$

Root Test

$$\frac{3}{2}\sum_{k=1}^{\infty} \sqrt[k]{\left(\frac{6}{k}\right)^k}$$

$$\lim_{k\to\infty} \frac{6}{k} \rightarrow 0$$

$0 < 1$

Convergent

27. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

AST

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \rightarrow \frac{1/n}{1/2\sqrt{n}} = \frac{2}{\sqrt{n}} = 0$$

$$n^{1/2} = \frac{1}{2}n^{-1/2}$$

$$\frac{1}{n}\cdot\frac{2\sqrt{n}}{1} = \frac{2\sqrt{n}}{n}$$

27. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

AST → Convergent

$$f(x) = \frac{\ln x}{\sqrt{x}} =$$

$$f'(x) = \frac{\left(\frac{1}{x}\right)\left(x^{1/2}\right) - \left(\frac{2}{x^{1/2}}\right)(\ln x)}{x}$$

$\ln x \to 1/x$

$\sqrt{x} \to 1/2x^{1/2}$

$$f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$$

$2 - \ln x < 0$

$e^{\ln x} > e^2$

$x > e^2$

31. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

$a_n = \tan\left(\frac{1}{n}\right)$ Diverges

$b_n = \dfrac{1}{n}$ Diverges

$$\lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \frac{\sec^2(1/n)(-1/n^2)}{(-1/n^2)}$$

$$\lim_{n\to\infty} \sec^2(1/n) \rightarrow 1 > 0$$

Limit compare Test

36. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

$\frac{a_{n+1}}{a_n}$ $\frac{1}{5} < 1$

Ratio test Converges

$$\frac{\frac{(n+1)^2+1}{5^{n+1}}}{\frac{n^2+1}{5^n}} = \frac{n^2+2n+2}{5^n \cdot 5} \cdot \frac{5^n}{n^2+1}$$

$$\rightarrow \frac{1}{5}\left(\frac{n^2+2n+2}{n^2+1}\right) = \frac{1}{5}$$

41. $$\sum_{k=1}^{\infty} \frac{5^k / 4^k}{3^k / 4^k + 4^k / 4^k} = \frac{\left(5/4\right)^k}{\left(3/4\right)^k + 1}$$

$$\lim_{k \to \infty} \frac{\left(5/4\right)^k}{\left(3/4\right)^4 + 1} = \frac{\infty}{1} = \infty$$

Divergant

Test for Divergance

41. $$\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$$

$$\frac{5^k}{3^k+4^k} < \left(\frac{5}{4}\right)^k \quad \text{LCT}$$