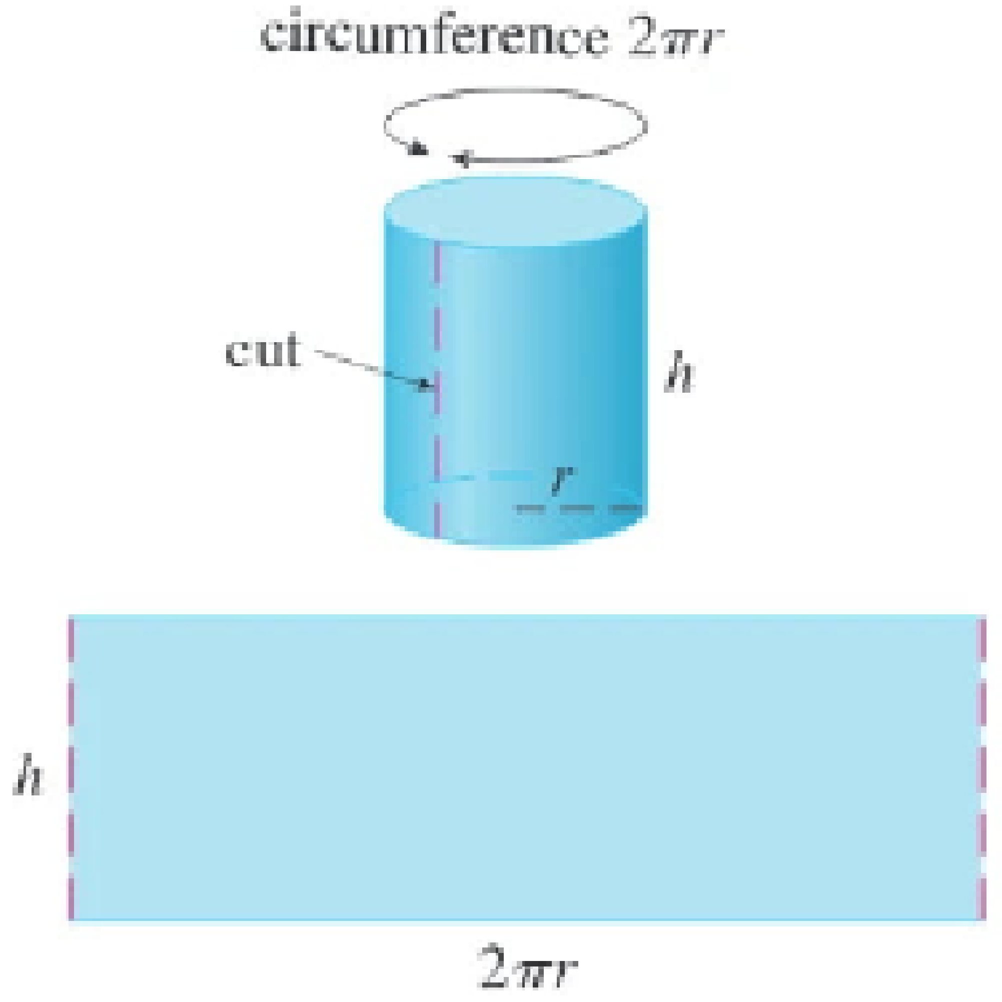
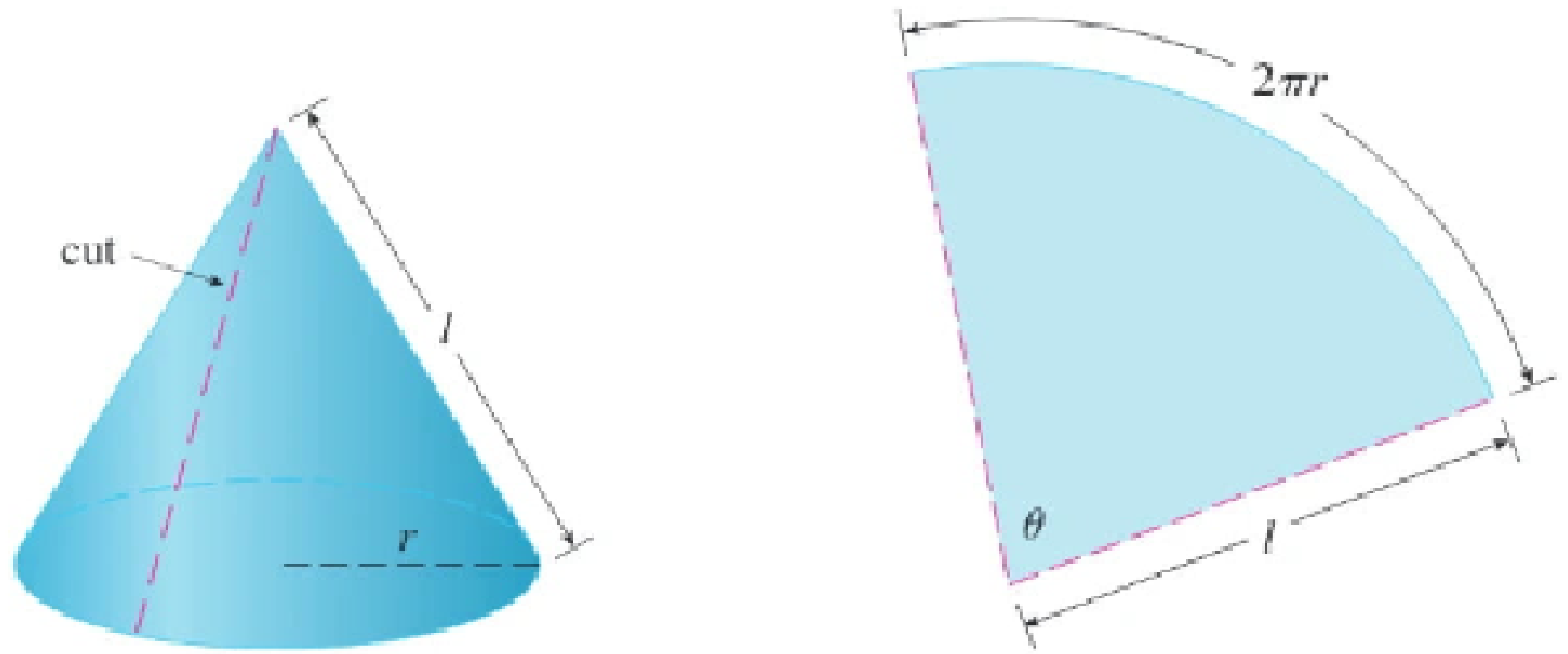
Section 8-2
Surface Area

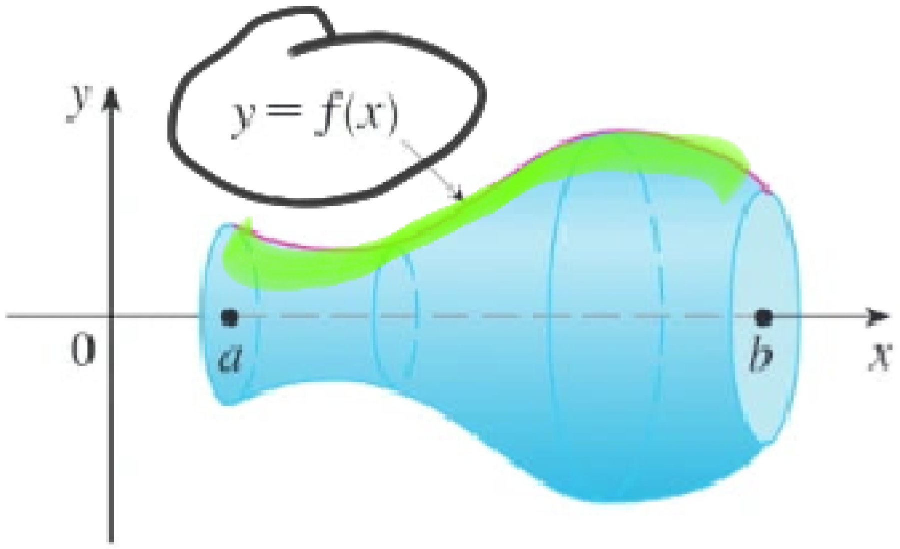




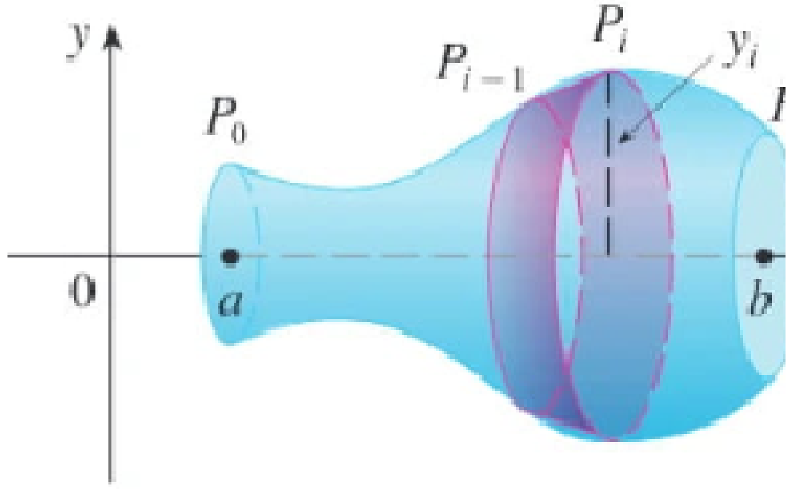
arc length

$$S = \int_a^b \boxed{2\pi f(x)} \boxed{\sqrt{1 + [f'(x)]^2}\, dx}$$

Circumfrence



(a) Surface of revolution

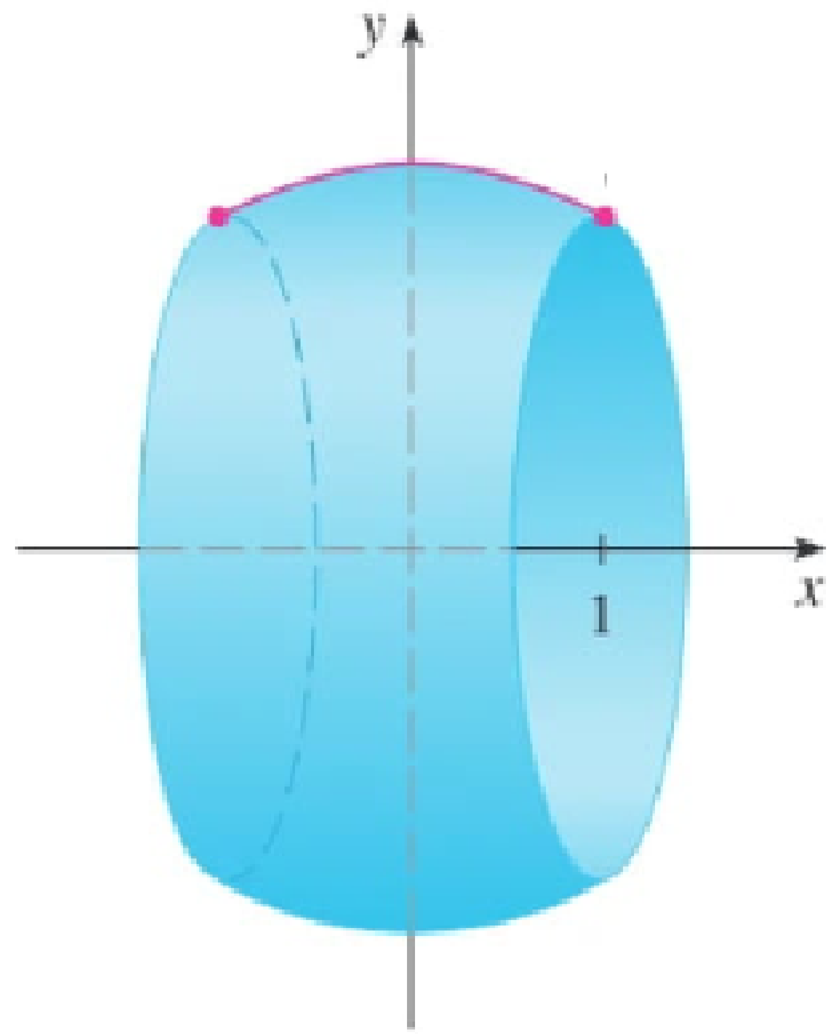


(b) Approximating band

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

$$C = 2\pi r$$

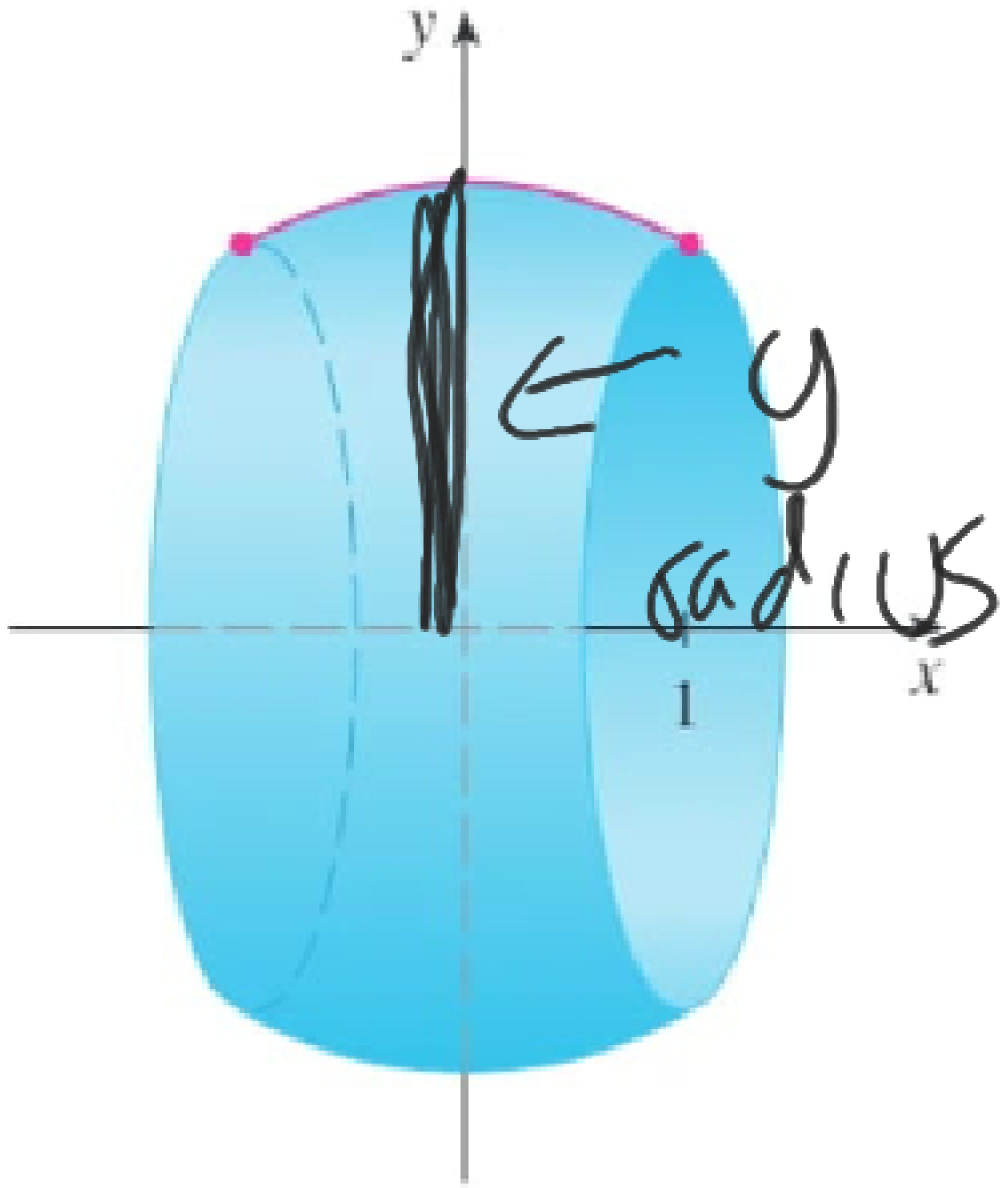
EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)



$$y = \sqrt{4 - x^2} \qquad -1 \leq x < 1$$

$$\int_a^b 2\pi y \sqrt{1 + (y')^2}\, dx$$

EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)



$$2\int_0^1 2\pi\left(\sqrt{4-x^2}\right)\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}$$

$$y = \sqrt{4-x^2} \qquad (4-x^2)^{1/2}$$

$$y' = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

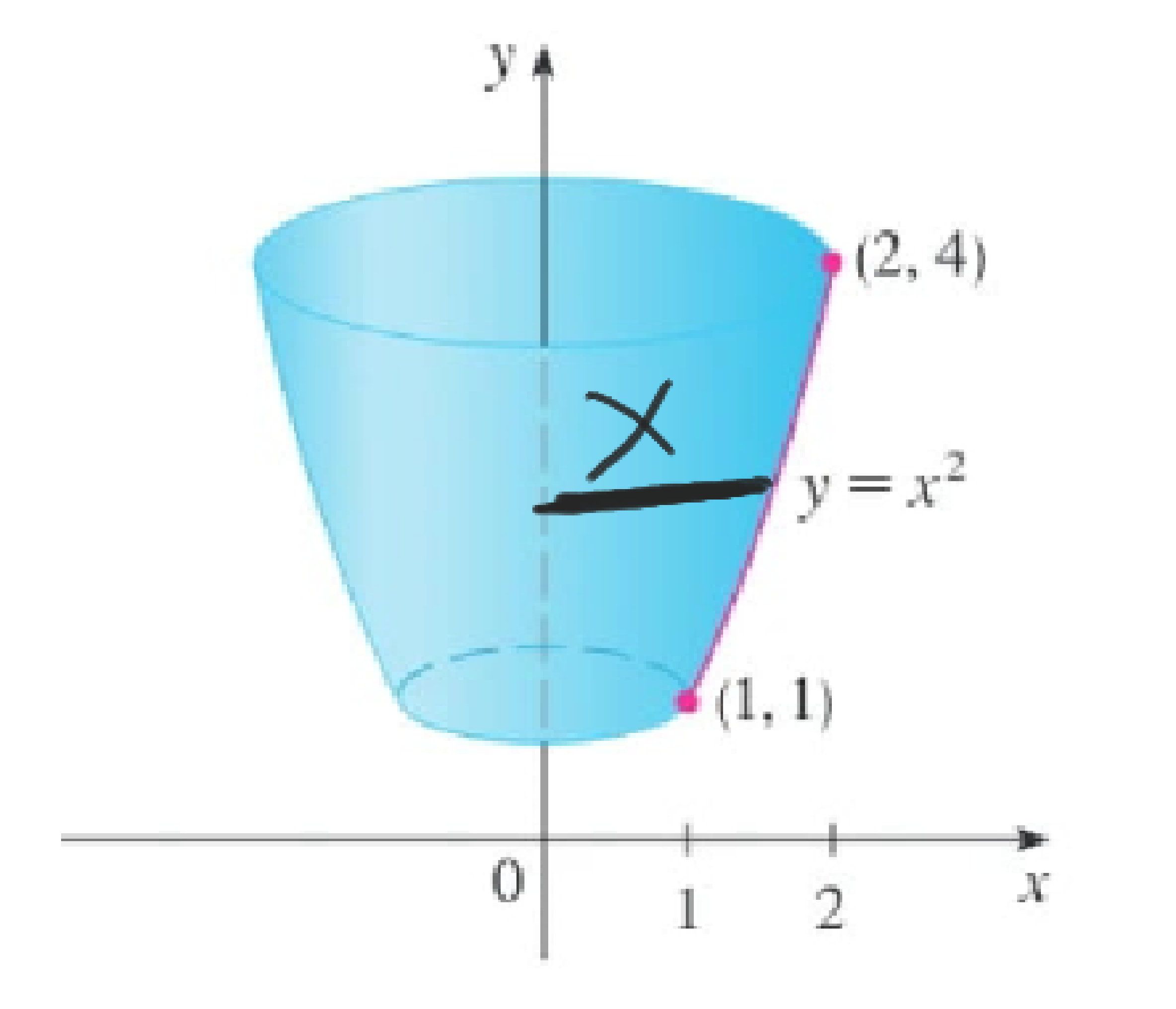
$$2\int_0^1 2\pi\left(\sqrt{4-x^2}\right)\boxed{\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}}$$

$$1+\frac{x^2}{4-x^2}=\frac{4-x^2+x^2}{4-x^2}=\sqrt{\frac{4}{4-x^2}}=\frac{2}{\sqrt{4-x^2}}$$

$$4\pi\int_0^1 \cancel{\left(\sqrt{4-x^2}\right)}\left(\frac{2}{\cancel{\sqrt{4-x^2}}}\right)dx=8\pi\int_0^1 1\,dx$$

$$8\pi\left(x\right)\Big|_0^1=\boxed{8\pi}$$

EXAMPLE 2 The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated
y-axis. Find the area of the resulting surface.



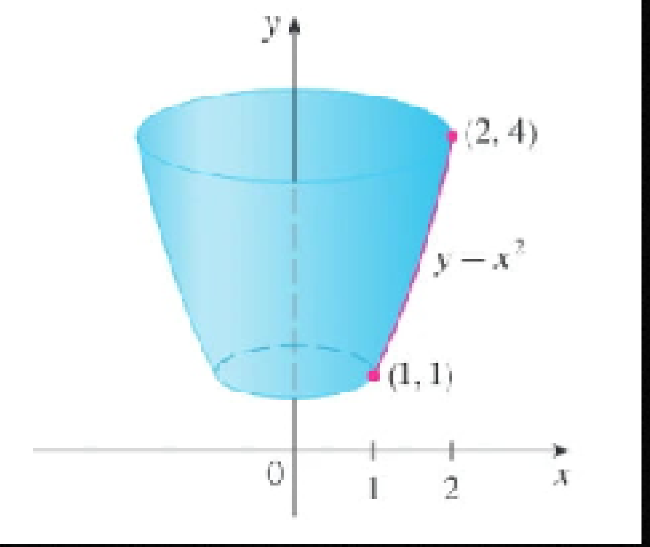
$y = x^2 \quad (1,1) \text{ to } (2,4)$

y-axis

$$\int_1^2 2\pi x \sqrt{1 + (2x)^2}\, dx$$

$y' = 2x$

EXAMPLE 2 The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.



$$\int_1^2 2\pi x \sqrt{1 + (2x)^2}\, dx \qquad u = 1 + 4x^2$$

$$2\pi \int_1^2 x \sqrt{1 + 4x^2}\, dx \qquad du = 8x\, dx$$

$$\tfrac{1}{8}\, du = x\, dx$$

$$2\pi \int_1^2 \frac{1}{8} u^{1/2}\, du = \frac{\pi}{4}\left[\frac{2}{3} u^{3/2}\right]$$

$$\frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right] \longrightarrow \frac{\pi}{6}\left[\left(1+4x^2\right)^{3/2}\right]_1^2$$

$$\boxed{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

EXAMPLE 3 Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$, about the x-axis.

$y = e^x \quad 0 \leq x \leq 1 \quad x\text{-axis}$

$$\int_0^1 2\pi (e^x)\sqrt{1+e^{2x}}\,dx$$

EXAMPLE 3 Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$, about the x-axis.

$x = \ln y \qquad 1 \le y \le e \qquad$ x-axis

$x' = \frac{1}{y}$

$u = 1 + \frac{1}{y^2}$

$\int_1^e 2\pi y \sqrt{1 + \frac{1}{y^2}}\, dy \qquad du = -2y^{-3}$

$$\overset{x\text{-axis}}{\int 2\pi y \, ds} \quad \text{or} \quad \overset{y\text{-axis}}{\int 2\pi x \, ds}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad y = ?$$

or

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad x = ?$$

$$y=\frac{x^3}{6}+\frac{1}{2x},\ \frac{1}{2}\leqslant x\leqslant 1$$

$$y'=\frac{x^2}{2}-\frac{1}{2x^2}$$

$$\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2$$

$$1+\left(\frac{x^4-1}{2x^2}\right)^2=\frac{x^8-2x^4+1}{4x^4}+\frac{4x^4}{4x^4}$$

$$\frac{x^8+2x^4+1}{4x^4}=\sqrt{\frac{(x^4+1)^2}{4x^4}}=\frac{x^4+1}{2x^2}$$

$$2\pi\int_{1/2}^{1}\left(\frac{x^3}{6}+\frac{1}{2x}\right)\left(\frac{x^4+1}{2x^2}\right)dx$$

$$\left(\frac{2x^4+6}{12x}\right)\left(\frac{x^4+1}{2x^2}\right)$$

$$8x^4$$

$$\frac{2x^8+2x^4+6x^4+6}{24x^3}=\frac{x^5}{12}+\frac{x}{3}+\frac{1}{4x^3}$$

$$2\pi\int_{1/2}^{1} \frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}$$

$$x^{-3}$$

$$2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_{1/2}^{1}$$

$$-\frac{1}{2}x^{-2}$$

$$2\pi\left[\left(\frac{1}{72} + \frac{1}{6} - \frac{1}{8}\right) - \left(\frac{1}{4608} + \frac{1}{24} - \frac{1}{2}\right)\right]$$