Lecture Notes for Chapter 8-1 and 8-2

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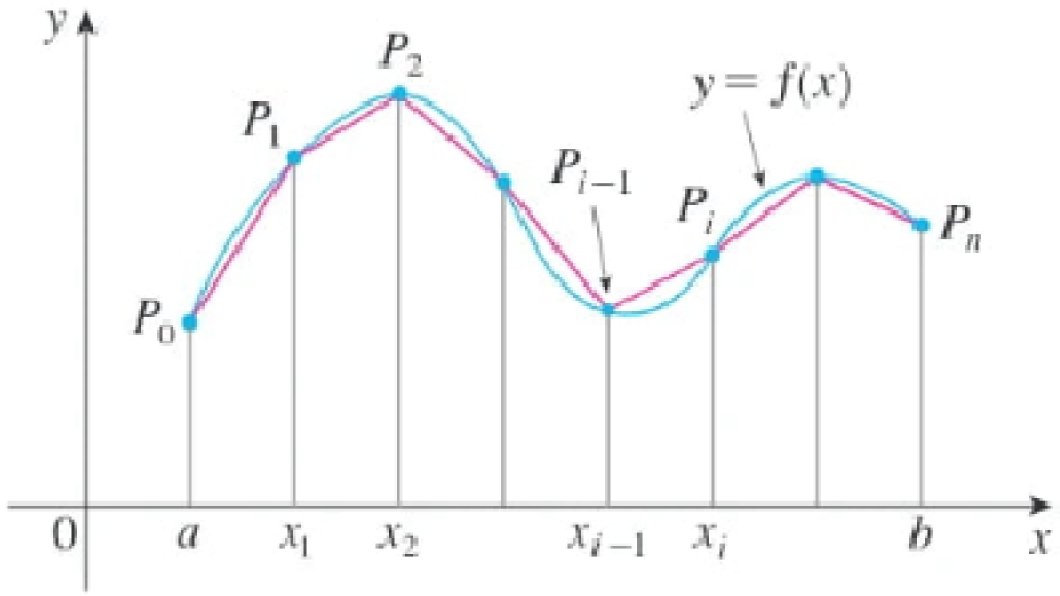
Three Rivers Community College
Calculus Early Trans 8th Edition Chapter 8-1 and 8-2

Key Topics
Arc Length
The Arc Length Function
Area of a Surface of Revolution

Arc Length

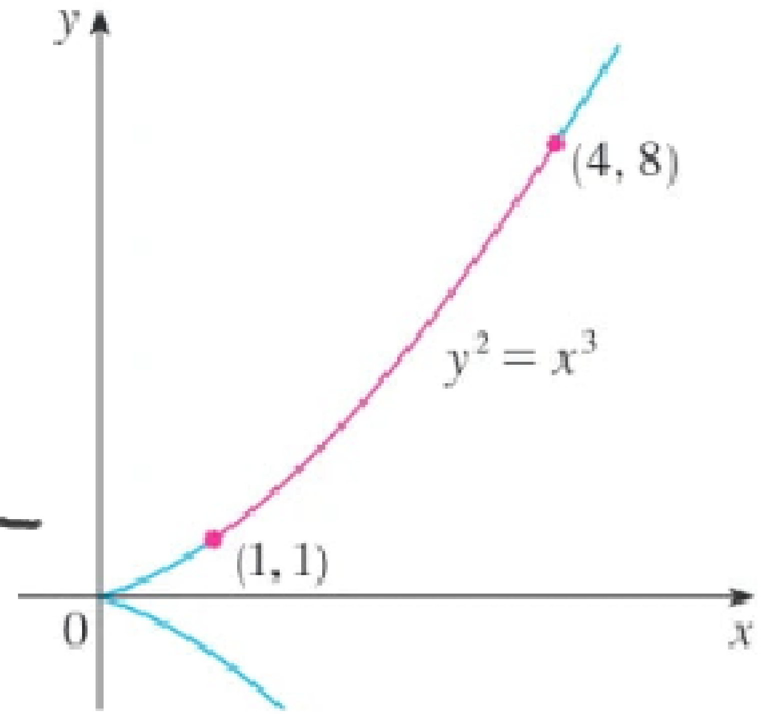


2 The Arc Length Formula If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$. (See Figure 5.)



$y^2 = x^3$

$y = x^{3/2}$

$\int_c^b \sqrt{1 + f'(x)^2}\, dx$

$f(x) = x^{3/2}$

$f'(x) = 3/2\, x^{1/2}$

$\int_1^4 \sqrt{1 + (3/2\, x^{1/2})^2}\, dx$

$\int_1^4 \sqrt{1 + 9/4\, x}\, dx$

$u = 1 + 9/4\, x \quad du = 9/4\, dx \quad x = 1 \quad u = 13/4 \quad x = 4 \quad u = 10$

$$\int_{1}^{4} \sqrt{1 + 9/4\, x}\; dx$$

$u = 1 + 9/4\, x$

$du = 9/4\, x$

$x = 1 \quad u = 13/4$

$x = 4 \quad u = 10$

$$\frac{4}{9} \int_{13/4}^{10} u^{1/2}\, du$$

$$\frac{4}{9} \left[\frac{2}{3} u^{3/2} \right]_{13/4}^{10} = \boxed{\frac{8}{27} \left(10^{3/2} - (13/4)^{3/2} \right)}$$

$$\left(1 + 9/4\, x\right) \Big]_{1}^{4}$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

$$\int_a^b \sqrt{1+(f'(y))^2}\,dy$$

$$x = y^2$$

$$f'(y) = 2y$$

$$\int_0^1 \sqrt{1+4y^2}\,dy$$

$$y = \frac{1}{2}\tan\theta$$

$$dy = \frac{1}{2}\sec^2\theta\,d\theta$$

$$\sqrt{1+4(1/2\tan\theta)^2}$$

$$\sqrt{1+\tan^2\theta}$$

$$\sqrt{\sec^2\theta} = \sec\theta$$

$y = 0 \quad 0 = 1/2\tan\theta \quad \theta = 0$

$y = 1 \quad 1 = 1/2\tan\theta$

$2 = \tan\theta$

$\tan^{-1}2 = \theta = a$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$$\sqrt{1 + 4y^2}$$

$$\sqrt{1 + \tan^2\theta}$$

$$(2y)^2$$

$$(2y)^2 = \tan^2\theta$$

$$2y = \tan\theta$$

$$y = 1/2 \tan\theta$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

$$\int_0^1 \sqrt{1+4y^2}\, dy$$

$a = \tan^{-1} 2$

$$\int_0^a \sec\theta \cdot \frac{1}{2}\sec^2\theta \, d\theta$$

$uv - \int v\, du$

$$\frac{1}{2}\int_0^a \sec^3\theta \, d\theta$$

$u = \sec\theta \quad v = \tan\theta$

$du = \sec\theta\tan\theta \quad dv = \sec^2\theta\, d\theta$

$$\frac{1}{2}\int_0^a \sec\theta \ \sec^2\theta \, d\theta$$

$$\frac{1}{2}\left[\sec\theta\tan\theta - \int \sec\theta\tan^2\theta \, d\theta\right]$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

$$\frac{1}{2}\int_0^a \sec^3\theta\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta - \int \sec^3\theta - \int \sec\theta\right]$$

$$\boxed{\int_0^a \sec^3\theta\, d\theta} = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

$$\frac{1}{2}\int_0^a \sec^3\theta\, d\theta = \frac{1}{4}\left(\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\Big|_0^a$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

$$\frac{1}{4}\left(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right)\Big|_0^a$$

$$\tan^{-1} 2 = a \rightarrow \tan a = 2$$

$$\frac{1}{4}\left(\sec a \tan a - \ln|\sec a + \tan a|\right)$$

$$\sec^2 a = 1 + \tan^2 a = 5 \qquad \sec a = \sqrt{5}$$

$$\boxed{\frac{1}{4}\left(2\sqrt{5} - \ln\left(\sqrt{5} + 2\right)\right)}$$

EXAMPLE 4 Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

The Arc Length Function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$y = x^2 - \tfrac{1}{8}\ln x$$

$$f'(x) = 2x - \frac{1}{8x}$$

$$\int_1^x \sqrt{1 + (2x - 1/8x)^2}\, dx$$

$$\int_1^x \left(2a + \frac{1}{8a}\right) da$$

$$a^2 + \frac{1}{8}\ln|a| \Big]_1^x$$

$$= \boxed{x^2 + \frac{1}{8}\ln x - 1}$$

$$1+\left(2x-\frac{1}{8x}\right)^2=1+4x^2-\frac{1}{4}-\frac{1}{4}+\frac{1}{64x^2}$$

$$=4x^2+\frac{1}{2}+\frac{1}{64x^2}$$

$$=\boxed{\left(2x+\frac{1}{8x}\right)^2}$$

$$4x^2+\frac{1}{4}+\frac{1}{4}+\frac{1}{64x^2}$$

$$\sqrt{\left(2x+\frac{1}{8x}\right)^2}=2x+\frac{1}{8x}$$