Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \to \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s$$

or

$$\sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Example 2

Show that the series $\sum_{n=1}^{\infty} \boxed{\frac{1}{n(n+1)}}$ is convergent, and find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(1 \cancel{- \frac{1}{2}}\right) \cancel{+ \left(\frac{1}{2}\right.} \left.\cancel{- \frac{1}{3}}\right) \cancel{+ \left(\frac{1}{3}\right.} \left. - \frac{1}{4}\right)$$

$$= 1 - \frac{1}{n+1} = \boxed{1}$$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \boxed{\frac{a}{1-r}} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1-\left(-\frac{2}{3}\right)} =$$

$a = 5$

$$r = -\frac{2}{3}$$

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$$

Convergent $|r| < 1$ to $\boxed{3}$

$\left|-\frac{2}{3}\right|$

Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

$$\sum_{n=1}^{\infty} a\,r^{\boxed{n-1}}$$

$$\boxed{2^{2n} \rightarrow 4\left(4^{n-1}\right)}$$

$$3^{1-n} \rightarrow \frac{1}{3^{n-1}}$$

$$\frac{4\left(4^{n-1}\right)}{3^{n-1}}$$

$$\boxed{4\left(\frac{4}{3}\right)^{n-1}}$$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} \rightarrow ar^{n-1}$

$$2^{2n-1} = \frac{2^{2n}}{2} = \frac{4^n}{2} \rightarrow \frac{4(4^{n-1})}{2}$$

$$\frac{\boxed{6 \cdot 4}\,\boxed{(4^{n-1})}}{\boxed{3}\,\boxed{(3^{n-1})}\cdot\boxed{2}} = \boxed{4}\,\boxed{\left(\frac{4}{3}\right)^{n-1}}$$

A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in $\mathrm{mg/mL}$) after the injection on the n th day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by $0.2\ \mathrm{mg/mL}$.

$$C_1 \rightarrow 0.2$$

$$C_2 \rightarrow 0.2 + (0.3)(0.2) = 0.26$$

$$C_3 \rightarrow 0.2 + \boxed{(0.3)}\left(\boxed{0.26}\right) = 0.278$$

$$0.2 + \left(0.2 + (0.3)(0.2)\right)(0.3)$$

$$0.2 + 0.06 + (0.3)^2(0.2)$$

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \cdots + \boxed{0.2(0.3)^{n-1}}$$

$$C_n = 0.2(1 - 0.3^{n-1})$$

$$= \frac{0.2(1-0.3^n)}{1-0.3} = \frac{0.2}{0.7}(1-0.3^n)$$

$$= \boxed{\frac{2}{7}} \qquad = \frac{2}{7}\boxed{(1-0.3^n)}$$

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

7 Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Harmonic Series

$$s_{2^n} > 1 + \frac{n}{2}$$

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} \frac{n^2 / n^2}{5n^2+4 / n^2} = \frac{1}{5+4/n^2} \Rightarrow \frac{1}{5}$$

$$\neq 0$$

Test for Divergance

$\lim_{n \to \infty} a_n \neq 0$ $\quad \sum a_n$ Divergrs

Geometric Series

$|r| < 1$ Converges

$|r| \geq 1$ Diverges

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

(i)

$$\sum_{n-1}^{\infty} ca_n = c \sum_{n-1}^{\infty} a_n$$

(ii)

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(iii)

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$. $=3+1=\boxed{4}$

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 3(1) = 3$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n} \Rightarrow \left(\frac{1}{2}\right)^n \rightarrow \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = 1$$

$$r=\frac{1}{2}<1 \quad \frac{a}{1-r} \quad \frac{1/2}{1-1/2}$$

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 1

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.